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On Spatial Birth-Death and Matching Processes, and Poisson Shot-Noise Fields

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On Spatial Birth-Death and Matching Processes, and Poisson Shot-Noise Fields

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Dedicated to my parents.

On Spatial Birth-Death and Matching Processes, and Poisson Shot-Noise Fields

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In this dissertation we deal with certain spatial stochastic processes that are closely related to Spatial birth-death (SBD) processes. These processes are stochastic processes that model the time-evolution of interacting individuals in a population, where the interaction between individuals depends on their relative locations in space. In this dissertaion, we consider three models of such processes with births and deaths that are amenable to long-term analysis. A common feature of all these models is that the particles in the system interact at a distance of one.

In Chapter 2, we introduce and examine a hard-core stochastic spatial point process with births and deaths. In this process, the arrival of particles is modeled using a Poisson point process in space and time. The process evolves according to the following interaction scheme. A newly arriving point "kills" each conflicting points independently with probability $\rho \in [0, 1]$. Two points are conflicting if they are within a distance one from each other. The new particle is accepted into the system

if all conflicting points are killed and removed. A particle present in the system stays in the system until it is removed due to interaction with a conflicting new arrival. Such stochastic models have been studied earlier for modeling populations of interacting individuals or as spatial queuing and resource sharing networks. We construct this process on the whole Euclidean space $\mathbb{R}^d$, $d \in \mathbb{N}^+$. If ρ is large enough, we show existence of a stationary regime, and also show exponential convergence to the corresponding stationary distribution.

In Chapter 3, we apply the techniques developed in the previous chapter to the Glauber dynamics of the Gibbs hard-core model in the infinite Euclidean space $\mathbb{R}^d$. This is a stochastic process where particles arrive according to a space-time Poisson point process of intensity $\lambda > 0$. An incoming particle is accepted into the system if there is no particle already in the system within a distance one from it. Further, every accepted particle departs at a constant rate 1. We give a bound on λ for existence of a stationary regime. We survey other probabilistic and function analytic techniques in the literature for the existence of a stationary regime, and compare the corresponding bounds obtained on λ.

In Chapter 4, we describe a process where two types of particles, marked by the colors *red* and *blue*, arrive in a domain D at a constant rate and are to be matched to each other according to the following scheme. At the time of arrival of a particle, if there are particles in the system of opposite color within a distance one from the new particle, then, among these particles, it matches to the one that had arrived the earliest. In this case, both the matched particles are removed from the system. Otherwise, if there are no particles within a distance one at the time of the arrival, the particle gets added to the system and stays there until it matches with another point later. Additionally, a particle departs from the system on its own at a constant rate $\mu > 0$, due to a loss of patience. We study this process both when D is a compact metric space and when it is a Euclidean domain, $\mathbb{R}^d$, $d \geq 1$.

When D is compact, we give a product form characterization of the steady state probability distribution of the process. We also prove an FKG type inequality, which establishes certain clustering properties of the red and the blue particles in the steady state. When D is the whole Euclidean space, we use the time-ergodicity of the construction scheme to construct a stationary regime.

In addition, in Chapter 5, we develop a Sanov-type Large Deviation Principle (LDP) for dense Poisson point processes on compact domains of $\mathbb{R}^d$. We then transform this result to obtain an LDP for Poisson shot-noise fields. Shot-noise fields show up in several real-world applications, and we motivate our results using applications to the modeling of dense wireless communication networks.

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Chapter 1

Introduction

In this thesis, we mainly deal with three examples of stochastic processes whose state space is a collection of particles living in a subset of a Euclidean space $\mathbb{R}^d$, $d > 0$, and each process evolves as particles arrive and depart in the system. In each of these examples, we are interested in the study of the long-term behavior of these system and characterize their steady-state if it exists. Additionally, we present a Large Deviation Principle (LDP) for dense Poisson point processes, and a related LDP for the shot-noise they generate with respect to a response function.

Spatial models stochastic models with births and deaths have received increasing attention as models for time-evolution of interacting individuals in a population. They find applications in a variety of fields, from physics to wireless communication to social sciences. In these models, the interactions between individuals crucially depends on the location of the individuals in space – for instance, one may want to account for factors such as overcrowding, which may decrease the birth rate of a particle at a point and increases the death rate of a particle. An important aspect of studying these system is to understand their long-run behavior, whether they are stable and exhibit a stationary regime. In Chapters 2 and 4, we will introduce two new models that are amenable to the study of their long-run behavior, and in Chapter 3, we consider a classical spatial birth-death process and survey the techniques used to study its long-run behavior.

The first model we deal with, which we call the *Hard-core spatial process with births and deaths* (Hard-core SwBD), is introduced in Chapter 2. It is a continuous time Markov process that can be described informally as follows. Imagine particles that have a hard-core radius of 1/2, so that no two particles can be present within a distance of 1 from each other simultaneously in the system at anytime. Suppose that these particles arrive uniformly at random in a domain. Whenever a new particle arrives into the system, it may be conflicting with some points already present in the system. We propose a mechanism where each of the conflicting particles is removed using with certain probability, and the new point is accepted into the system if no conflicting particles are present in the system. In this problem, we give a construction of a process with these dynamics in the infinite Euclidean domain. This construction is used later in the construction of a stationary regime, under certain sufficient conditions.

In Chapter 3, we consider the Glauber dynamics of the Gibbs hard-core model, that is also a spatial birth-death process of great importance in statistical physics. Here, we again have the hard-core condition that no two particles can be within in a distance of 1 from each other at any time. The process evolves as new particles arrive at a constant rate, uniformly randomly into the system; new particles are accepted if there are no conflicting particles in the system; and once accepted, they depart at a constant rate 1. For this process, we use the techniques developed in Chapter 2, to give sufficient conditions for the existence of a stationary regime. We then compare similar results present in the literature on the existence of stationary regime, that utilize different methodologies.

Chapter 4 develops a new model of population dynamics that is inspired by the matchings that occur in a ride-sharing network, such as Uber or Lyft. Here, we consider a continuous time Markov process where there are two types of particles. Each type of particle arrive at a constant rate and uniformly randomly on the domain, and matches to the oldest particle of the opposite type present in the system, that is within a distance one from it. If such a match is made, both the particles are removed from the system, while if there is no such match available, the incoming particle is added to the system. In addition, all accepted particles are removed from the system at a constant rate 1. We study this process both in the bounded and the infinite Euclidean domain $\mathbb{R}^d$. On the bounded domain, we are able to characterize the stationary distribution as a product form distribution. Extracting any quantitative geometric or clustering properties from this product form distribution requires computing the normalizing constant, which is a computationally hard problem. Instead, we prove that the steady state distribution satisfies an FKG type log-submodularity condition, that show qualitatively that the particles of the same type are more clustered than the particles in a Poisson point process. On the infinite Euclidean domain, we are again able to establish existence and uniqueness of a stationary regime.

In Chapter 5, using techniques from large deviations theory, we obtain a Sanov-type LDP for the dense Poisson point processes on a compact domain of $\mathbb{R}^d$. Then, using the contraction principle, we transform the LDP to obtain an LDP for shot-noise processes with respect to a response function. The motivation for this work comes from the need to study the occurrence of rare events in a dense wireless communication network, where the transmission nodes are modeled as points in a point process [53].

1.1 Preliminaries

In the following sections, we will present an overview of the notation and concepts required in the later chapters. In Section 1.1.1 we present a brief overview on random measures and point processes, which provide the language to express the problems that interest us. Spatial birth-death processes are a class of stochastic processes that are closely related to stochastic processes in this thesis, and we borrow a large amount of concepts and terminology from them. Section 1.1.2 records the basic definitions and foundational references in the theory of spatial birth-death processes. Finally, in Section 1.1.3 we describe a sufficiently large family of Gibbs measures in the continuum, and the specify the associated Glauber dynamics.

1.1.1 Random Measures and Point Processes

For a detailed presentation on the theory of Random Measures and Point Processes, we refer the reader to the following classical books: [42, 17, 16]. More recent expositions can be found in [4, 48, 67]. Applications to the modeling of wireless networks can be found in [5].

Framework: Let $(D, \mathcal{D})$ be a locally compact, second countable, Hausdorff topological space (abbreviated as l.c.s.h.). Such a space is also Polish, i.e., it is completely metrizable, such that the topology induced by the metric is separable.

Consider the measure space $(D, \mathcal{B}(D))$, where $\mathcal{B}(D)$ is the Borel σ-algebra associated with the topology $\mathcal{D}$. We let $\mathcal{B}_c(D)$ denote the set of relatively compact sets in $\mathcal{B}$. We denote by

1. $\mathcal{G}(D)$: the class of measurable function $f : D \to \mathbb{R}$,

2. $\mathcal{G}_b(D) \subset \mathcal{G}(D)$: the class of bounded measurable functions,

3. $\mathcal{G}_+(D) \subset \mathcal{G}(D)$: the class of bounded measurable functions $f : D \to \mathbb{R}^+$,

4. $\mathcal{G}_c(D) \subset \mathcal{G}(D)$: The class of continuous functions with compact support.

We may drop the reference to the space D if it is clear from the context. In our work, we usually deal with spaces D that are subsets of the Euclidean space $\mathbb{R}^d$, for some $d > 0$. Here, we use the usual topology on D induced by the Euclidean metric on $\mathbb{R}^d$.

A *Radon measure* μ on $(D, \mathcal{B})$ is a measure such that $\mu(B) < \infty$ for all $B \in \mathcal{B}_c$. We denote the space of Radon measures on D by $M(D)$, and let $\bar{\mathcal{M}}(D)$ denote the σ-algebra on $\bar{M}(D)$ generated by the maps $N_B : \mu \mapsto \mu(B)$, $B \in \mathcal{B}$. For any measurable function f on D, we define

$$\mu(f) := \int_D f(s)\mu(\mathrm{d}s),$$

when the integral is well-defined in the Lebesgue sense. Further, we let $\tilde{M}(D) \subset \bar{M}(D)$ denote the space of locally finite counting measures on D, i.e., $\mu \in \tilde{M}(D)$ if and only if $\mu(B) \in \mathbb{N}$ for all $B \in \mathcal{B}_c$. Finally, we let $M(D) \subset \tilde{M}(D)$ denote the space of simple counting measures, which satisfy the property that

$$M(D) := \{\mu \in \tilde{M}(D) : \mu(\{x\}) \leq 1 \ \forall x \in D\}.$$

For $\mu \in M(D)$ we will often abuse notation and let μ refer also to its support, $\operatorname{supp}\mu$. We denote by $\tilde{\mathcal{M}}(D)$ and $\mathcal{M}(D)$ the σ-algebras induced by $\mathcal{M}(D)$ on $\tilde{M}(D)$ and $M(D)$ respectively. Further, for any $A \subset D$, we will let $\mathcal{M}_A(D)$ denote the σ-algebra generated by functions $N_B : \mu \mapsto \mu(B)$, $B \subset A$.

For any measure $\mu \in \bar{M}(D)$ and $\Lambda \subseteq D$, we let $\mu_\Lambda \in \bar{M}(D)$ defined by $\mu_\Lambda(B) = \mu(B \cap \Lambda)$, for all $B \in \mathcal{B}(D)$.

A *random measure* is a random element in $(\bar{M}(D), \bar{\mathcal{M}}(D))$, i.e., it is a measurable map from a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ to $(\bar{M}(D), \bar{\mathcal{M}}(D))$. We shall call a random measure Φ a *point process* if $\Phi \in \tilde{M}(D)$ a.s., and a *simple point process* if $\Phi \in M(D)$ a.s.

Poisson Point Process: Let ν be a Radon measure on D. A *Poisson point process* (PPP) Φ on a D, with intensity measure ν, is a random measure on D as defined above, with the following properties.

1. For all bounded $A \in \mathcal{B}_c$, $\Phi(A)$ is a Poisson random variable with mean $\nu(A)$ and

2. For any finite collection of disjoint bounded Borel subsets $A_1, \ldots, A_n$, $\Phi(A_i)$, $1 \leq i \leq n$, are independent random variables.

If D is a Borel measurable subset of $\mathbb{R}^d$, then a Homogeneous PPP on D with intensity λ, $\lambda \in \mathbb{R}^+$, is a PPP with intensity measure $\lambda\ell(\cdot)$, where ℓ is the Lebesgue measure on D.

Marked Point Process: Let D and $\mathbb{K}$ be two l.c.s.h. spaces. A point process $\bar{\Phi}$ on $D \times \mathbb{K}$ is called a marked point process on D with marks in $\mathbb{K}$, if $\hat{\Phi}(B \times \mathbb{K}) < \infty$ a.s. for all $B \in \mathcal{B}_c$. Its projection on D, i.e., $\Phi(\cdot) = \hat{\Phi}(\cdot \times \mathbb{K})$, is called the *ground point process.*

Moment Measures and Laplace Transform: Moment measures and Laplace transform are important objects that reveal information about the distribution of a point process. The *n-th order moment measure* is a measure on the product space D^n given by $m_\Phi^n(A_1 \ldots, A_n) = \mathrm{E}\prod_{i=1}^n \Phi(A_i)$. The first order moment measure of a point process is also referred to as its intensity measure. We will simply write m_Φ for m_Φ^1. In the following chapters, we will assume from now on that the intensity measure is σ-finite. For Poisson point processes, this definition matches with the earlier definition of the intensity measure.

The *Campbell averaging formula* will be extensively used later in the following chapters. It can be summarized by the observation that for any $f : D \to \mathbb{R}$, which is either non-negative or in $L^1(m_\Phi)$, the integral $\int_D f \mathrm{d}\Phi$ is well-defined random variable and its expectation is

$$\mathrm{E}\left[\int_D f \mathrm{d}\Phi\right] = \int_D f \mathrm{d}m_\Phi.$$

The *Laplace transform* plays an important role for random measures and point processes, in particular because it completely characterizes the probability distribution. The Laplace transform is a functional defined on the set of all measurable functions $f : D \to \bar{\mathbb{R}}_+$ by

$$\mathcal{L}_\Phi(f) = \mathrm{E}\left[\exp\left(-\int_D f \mathrm{d}\Phi\right)\right]. \tag{1.1}$$

The Laplace transform of a PPP Φ on D with intensity measure m_Φ is given by

$$\mathcal{L}_\Phi(f) = \exp\left(-\int_D (1 - e^{-f}) \mathrm{d}m_\Phi\right), \tag{1.2}$$

for all $f : D \to \bar{\mathbb{R}}_+$.

Palm measures: Informally, a Palm measures of a point process Φ at a point $x \in D$ is the probability measure of Φ conditioned on having a point at location x. We shall define the Palm measure now. For a more detailed discussion on Palm measures, see [42].

To define the Palm measure, we first need the following definition. The *Campbell measure*, $\mathcal{C}_\Phi$, of a point process Φ is the unique σ-finite measure on $D \times M(D)$ characterized by

$$\mathcal{C}_\Phi(B \times L) = \mathrm{E}[\Phi(B)\mathbb{1}(\Phi \in L)] = \mathrm{E}\left[\int_B \mathbb{1}(\Phi \in L)\Phi(\mathrm{d}x)\right], \quad B \in \mathcal{B}, L \in \mathcal{M}(D).$$

Similarly, define the *reduced Campbell measure* to be

$$\mathcal{C}^!_\Phi(B \times L) = \mathrm{E}\left[\int_B \mathbb{1}(\Phi - \delta_x \in L)\Phi(\mathrm{d}x)\right].$$

We note that, for $B \in \mathcal{B}$, $\mathcal{C}_\Phi(B \times M(D)) = \mathrm{E}[\Phi(B)] = m_\Phi(B)$. Thus, by the measure disintegration theorem (see Theorem 15.3.3 of [42]), it follows that $\mathcal{C}_\Phi$ admits a disintegration

$$\mathcal{C}_\Phi(B \times L) = \int_B \mathrm{P}_\Phi^x(L) m_\Phi(\mathrm{d}x), \quad B \in \mathcal{B}, L \in \mathcal{M},$$

where P_Φ^x is a kernel from D to $M(D)$ that may be chosen to be a probability measure for m_Φ-almost all $x \in D$. For any $x \in D$, if P_Φ^x is a probability measure, then it is referred to as the *Palm measure* of Φ at x.

Papangelou Conditional Intensity: Informally, if Palm measure, P_Φ^x is the probability measure of Φ conditioned on finding a point at $x \in D$, the Papangelou conditional intensity is the density of finding a point at x conditioned on the state of the system.

If we assume that, for any $A \in \mathcal{B}_c$, the measure $\mathcal{C}_\Phi^!(A \times \cdot)$ is absolutely continuous with respect to the probability measure P, then the reduced Campbell measure disintegrates into

$$\mathcal{C}_\Phi^!(A \times L) = \int_L K(A, \eta)\, \mathrm{P}(\mathrm{d}\eta),$$

where K is called the *Papangelou kernel.* If K admits a density $\lambda^*(x, \eta)$ with respect to a reference measure ν (usually the Lebesgue measure or the intensity measure m_Φ) on D, then the density is called the *Papangelou conditional intensity.*

In this context, the Georgii-Nguyen-Zessin (GNZ) formula reads

$$\mathrm{E}\left[\int_D g(x, \Phi - \delta_x)\Phi(\mathrm{d}x)\right] = \mathrm{E}\left[\int_D \lambda(x, \Phi) g(x, \Phi) \nu(\mathrm{d}x)\right], \tag{1.3}$$

for all measurable $g : D \times M(D) \to \mathbb{R}$ such that at least one of the sides above is integrable.

Stationarity and Ergodicity: Let $D = \mathbb{R}^d$, for some $d > 0$. For $x \in \mathbb{R}^d$ and $\mu \in \bar{M}(\mathbb{R}^d)$, let $S_x\mu$ denote the measure defined by $S_x\mu(B) = \mu(B - x)$, $B \in \mathcal{B}(\mathbb{R}^d)$. We say that a point process Φ on $\mathbb{R}^d$ is stationary if, for all $x \in \mathbb{R}^d$, the measure $S_x\Phi$ has the same distribution as Φ. For a stationary point process there exists a constant $\lambda \geq 0$, such that the intensity measure $m_\Phi(B) = \lambda\ell(B)$ for all $B \in \mathcal{B}$. This constant λ is called the intensity of the point process. The Palm measure P_Φ^x of a stationary point process at $x \in \mathbb{R}^d$ is independent of the location x.

A point process is said to be ergodic if for all S_x invariant sets $L \in \mathcal{M}(\mathbb{R}^d)$, $\mathrm{P}(L)$ is equal to either 0 or 1.

1.1.2 Spatial Birth-Death Processes

Birth-death processes are popular in the queuing theory and operations research as models for population dynamics. These are continuous time Markov processes on $\mathbb{N}$, with the property that if the process is at state $n \in \mathbb{N}$, then it can only make jumps to either $n+1$ or $n-1$. Spatial birth-death (SBD) processes model the dependence of the birth and death rates of individuals on their locations in the system. For example, this dependence may be due factors such as overcrowding, which may decrease birth rate and increase death rate in real-world systems of interest.

An SBD process on $D \subset \mathbb{R}^d$, $d > 0$ is characterized as follows. The state space of the process is $M(D)$, the space of simple counting measures on D. The dynamics of the process are specified by giving the birth and death rate functions

$$b, d : D \times M(D) \to \mathbb{R}^+.$$

The interpretation of the functions b and d is that if the state of the Markov process is $\eta_t \in M(D)$, at time $t \in \mathbb{R}^+$, then a point is added to the system in a bounded measurable $A \subset D$, in the time interval $(t, t+\delta]$, with probability approximately $\delta \int_A b(x, \eta_t)\ell(\mathrm{d}x)$, and a point $y \in \eta_t$ is removed in $(t, t+\delta]$ with probability approximately $\delta d(y, \eta_t)$. The generator of the process described above takes the form

$$Lf(\eta) = \int_D b(x,\eta)[f(\eta \cup \{x\}) - f(\eta)]\mathrm{d}x + \int_D d(x,\eta)[f(\eta \backslash \{x\}) - f(\eta)]\eta(\mathrm{d}x).$$

Spatial birth-death processes were first discussed in by Preston in [64]. Under some conditions on the birth and death rate functions, b and d, Preston proved the existence and uniqueness of such processes on a bounded domain of the Euclidean space $\mathbb{R}^d$. The problem of convergence to a stationary state of this process was studied in [52, 59]. Problems of existence, construction and uniqueness on infinite real line were considered in [40]. Later, in [35] the authors gave probabilistic constructions of more general SBD processes as projections of higher dimensional Poisson point processes, and moreover, in [26, 33], the authors gave a coupling from the past construction of the stationary regime for some of these processes. In [8, 29, 27], the authors develop function analytic tools to study the existence and convergence to stationarity of SBD processes. More recently, in [7], the authors considered a certain SBD process, where the birth occur at a constant rate, while deaths occur due killing by some other particle establishing a connection at a random time. In the paper, the authors give an alternate construction of the process in the infinite domain, and a coupling from the past construction of the stationary regime. A number of coupling from the past constructions developed in this thesis are inspired from those in [7].

1.1.3 Gibbs Measures on Continuum

In this section we will define Gibbs probability measures on a Euclidean domain $D \subset \mathbb{R}^d$ associated with a finite pair potential of finite range. For more general definitions and a deeper exposition, see [37, 36] or Chapter 6 of [60]. We will define the Gibbs hard-core model in details later in Chapter 3.

In the following, let $D = \mathbb{R}^d$. For $x, y \in \mathbb{R}^d$, let $|x-y|$ denote the Euclidean distance between x and y. Let $\varphi : \mathbb{R}^d \to \mathbb{R}$ be a measurable even function. φ is called a pair potential. We assume that φ has finite range r, i.e., $\varphi(x) = 0$ if $|x| > r$.

Let $\Lambda \subset D$ be a bounded domain. The Hamiltonian $H_\Lambda : M(D) \to \mathbb{R}$ is given by

$$H_\Lambda(\eta) := \sum_{\substack{x,y\in\eta \\ \{x,y\}\cap\Lambda\neq\emptyset}} \varphi(x-y).$$

Ocasionally, we may also want to impose a boundary condition, i.e., fix the state outside the domain Λ, and see the effect on the Gibbs distribution. Such boundary conditions on Λ can be modeled as follows: for $\eta, \omega \in M(D)$ let $H_\Lambda^\omega(\eta) := H_\Lambda(\eta_\Lambda \omega_{\Lambda^c})$, where $\eta_\Lambda\omega_{\Lambda^c} := (\eta \cap \Lambda) \cup (\omega \cup \Lambda^c)$, and ω is the *boundary condition* imposed. Let P_Λ denote the Poisson point process on Λ with intensity 1:

$$P_\Lambda(\mathrm{d}x_1, \ldots, \mathrm{d}x_n) = \frac{e^{-\ell(\Lambda)}}{n!}\mathrm{d}x_1 \cdots \mathrm{d}x_n, \quad \forall n \in \mathbb{N},\ x_1, \ldots, x_n \in \Lambda.$$

Let $P_\Lambda^\omega(\mathrm{d}\eta) := \mathbb{1}_{\omega_{\Lambda^c}}(\eta_{\Lambda^c}) \times P_\Lambda(\mathrm{d}\eta_\Lambda)$ be a probability measure on $M(D)$, that fixes the point outside the domain Λ to ω_{Λ^c}, and inside Λ it is a Poisson point process.

The finite volume Gibbs measure on Λ at inverse temperature β, activity $\lambda > 0$ and boundary conditions ω is given by

$$\mu_\Lambda^\omega(\mathrm{d}\eta) := e^{\ell(\Lambda)}(Z_\Lambda^\omega)^{-1}\lambda^{|\eta|}\exp[-\beta H_\Lambda^\omega(\eta)]P_\Lambda^\omega(\mathrm{d}\eta), \tag{1.4}$$

where Z_Λ^ω is a normalizing constant. Hence, for all measurable functions f on $\mathcal{M}_\Lambda(D)$, we have

$$\mu_\Lambda^\omega(f) = (Z_\Lambda^\omega)^{-1}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\int_{\Lambda^n} e^{-\beta H_\Lambda^\omega(x)}f(x)\mathrm{d}x,$$

where we have identified the function on $\mathcal{M}_\Lambda(D)$ with the symmetric functions on $\cup_{n=0}^\infty \Lambda^n$.

For a given pair potential function, it is easy to see that the family of measures in (1.4) satisfies the so-called DLR compatibility conditions

$$\mu_\Lambda^\omega(\mu_V^{(\cdot)}(A)) = \mu_\Lambda^\omega(A), \quad A \in \mathcal{M}_\Lambda(D), \quad V \subset \Lambda \subset \mathbb{R}^d, \Lambda \in \mathcal{B}_c.$$

Thus, for $V \subset \Lambda$, $\mu_V^{\omega'_{\Lambda\setminus V}\omega_{\Lambda^c}}$ is the conditional distribution under μ_Λ^ω of the configuration inside V conditioned on the configuration $\omega'_{\Lambda\setminus V}$ on $\Lambda \setminus V$. Given a pair potential function, Gibbs measures on the infinite domain $D = \mathbb{R}^d$ are measures μ on $M(D)$ that satisfy the DLR condition locally:

$$\mu(A) = \mu(\mu_\Lambda^{(\cdot)}(A)), \quad \Lambda \in \mathcal{B}_c(\mathbb{R}^d),\ A \in \mathcal{M}_\Lambda(D). \tag{1.5}$$

Gibbs measures on bounded domains can also be characterized by their Papangelou conditional intensity (see [36] for this characterization on unbounded domains). The Papangelou conditional intensity of the Gibbs measure defined in (1.4) is

$$\lambda^*(x, \eta) = \lambda \exp\left[-\beta \sum_{y \in \eta} \varphi(x-y)\right], \tag{1.6}$$

where the sum is over a finite set of terms since φ is a finite range pair potential function.

The Glauber dynamics of a Gibbs measure is a SBD process, such that process is reversible with respect to the measure. We usually set the death rate, $d = 1$, and set the birth rate

$$b(x, \eta) = \lambda \exp\left[-\beta \sum_{y \in \eta} \varphi(x-y)\right].$$

Chapter 2

A Hard-core Spatial Stochastic Process with Simultaneous Births and Deaths[1]

2.1 Introduction

In this chapter, we develop a new class of spatial stochastic process with births and deaths. The process is studied on a domain $D \subseteq \mathbb{R}^d$, for some $d \geq 1$. The state space of the process is $M(D)$, the space of locally finite collection of points in D. In the process, new particles arrive at a constant rate 1 uniformly randomly in D. Each particle has a disk of radius $1/2$ attached to it. There is pairwise competition between a newly arriving particle and those particles already present in the system whose disks intersect the disk of the new particle. The interaction is such that the new arriving particle ”kills” an already accepted particle with probability ρ, independent of everything else. The new particle is then accepted if it manages to kill all the competing particles. Once accepted, a particle remains in the system until another arrival kills it. So, at any time, the state satisfies the hard-core condition that no two particles in it are within a distance 1 from each other. Further, in contrast with classical spatial birth-death (SBD) processes, multiple deaths can occur simultaneously. To refer to this process in this chapter we will call this process the *Hard-core spatial*

[1]The results and analysis presented in this chapter were originally published by the author in [54].

process with births and deaths (Hard-core SwBD) process.

2.2 Related Work

The Random sequential absorption (RSA) model can be seen as a limiting regime of this process as the limit $\rho \to 0$. The RSA model finds applications in the modeling of dynamic packing and scheduling protocols in operations research and wireless networking (see [25, 62, 61]). The RSA model have been used to study the performance of medium access and scheduling protocols in wireless networks [61], where a transmitting antenna blocks other antennas near it. The RSA model is especially difficult to study – even the jamming limit is difficult to calculate explicitly. It is therefore useful to study alternate models that approximate the RSA scheme. Our model also was also motivated by the applications to distribution of data over wireless communication networks, as a limiting regime of the model considered in [7]. In that model, particles arrive and are accepted into the system, and are removed as soon as they form a connection at a random time with any other particle in the system. We imagine that, in our model, connections are made instantly at the time of arrival.

Another classical spatial birth death processes that has been studied extensively is the Gibbs hard-core process (see [26, 8]), which is the subject of discussion in Chapter 3. This process also satisfies the hard-core condition that no two particles in the system can be within a distance 1 from each other at any time. This is a stochastic process where particles arrive according to a Poisson point process and incoming particles do not displace other particles ($\rho = 0$), but accepted particles may depart after waiting for a random time. It is conjectured that in this model, there is a phase transition that occurs as we increase the arrival rate $\lambda > 0$. For small values to λ there is a unique stationary regime, while for large values of λ the system is conjectured to have multiple stationary regimes. We conjecture that the hard-core SwBD process studied in this chapter also shows this behavior as we decrease ρ.

For point processes, moment measures give important average structural characteristic of the process, such as level of clustering or repulsion. A common approach used in the literature for the analysis of SBD processes is to look at the differential equations for moment measures ([28, 29]) obtained from the generator of the process. This typically yields an infinite family of coupled differential equations, one for moment measure for every order. To study the steady state, one can equate the time derivative to zero, which yields a hierarchical system of equations satisfied by the moment measures in this regime. The properties of the steady state can in principle be gleaned from these system of equations. While we refrain from following this approach, some parts of our analysis rely on the writing differential equations for moment measures of first order.

In the following, we will give a probabilistic construction of the process, and study its long-term behavior using this construction. On a bounded measurable domain D, the process can be constructed using the classical theory of jump Markov process. However, when $D = \mathbb{R}^d$, there is no "first jump" beyond time 0. That is, although locally the process looks like a jump process, in any short time interval there is an arrival somewhere in space, almost surely. One way to construct the process is to use the semigroup approach similar to the one in [46], where the authors give the construction for Glauber dynamics on infinite Euclidean space. Here, first, the form of the generator is obtain by extension from the generator on bounded domains. Then, its functional analytic properties are studied to argue the existence and uniqueness of a semigroup associated with the generator. In the following sections, we present an alternative construction, where the idea of construction for jump Markov processes is extended from bounded domains. In other related work, in [63] the author gives a framework for solving the existence problem for Glauber dynamics of interacting particle systems on infinite domains. This scheme can also be used to give a probabilistic construction of our process. However, this representation is not suitable for the analysis of the stationary regime. In our approach, we construct the process explicitly using a probability space that contains a marked Poisson point process on $\mathbb{R}^d \times \mathbb{R}^+$, that serves as the arrival process. The construction involves performing a thinning of this marked Poisson point process, using a backward investigation algorithm.

Using this construction, we also prove the existence of a stationary regime when ρ is large enough, with a lower bound depending on the dimension of the space. The technique utilized for showing this is based on a *coupling from the past* scheme inspired by [7]. This scheme bears resemblance to the Loynes' scheme for queuing systems (see e.g. [6]), where one constructs a stationary regime using a coupling of an infinite family of identical queuing systems, run using a common driving process, but starting with empty initial conditions from increasing negative times, $-T$, $T \in \mathbb{N}$. The limiting state at time 0, as $T \to \infty$, if it exists, is a stationary state. The key feature in our analysis is the use of differential equations for the density of discrepancies between two processes starting at two different times. We bound the rate of growth of the process and show that if ρ is large enough, these densities decay to zero exponentially with time. This is then shown to be sufficient for the coupling from the past algorithm to work. This establishes that for large enough values of ρ, the system has a unique stationary distribution.

In the following section, we begin with a formal definition of our model. Construction of this process for a constant rate of arrivals is discussed in Section 2.3. We then provide sufficient conditions for the coupling from the past argument to work in Section 2.4. The detailed construction of a stationary regime for the process is presented in Section 2.4.1. Finally, we conclude with the result showing that this process with arbitrary initial condition converges in distribution to the distribution of the process is stationary regime in Section 2.4.3.

2.3 General Formulation

We begin with a general description of a class of processes that we call *Hard-core Spatial processes with Births and Deaths* on bounded measurable Euclidean space $D \subset \mathbb{R}^d$. First we define

$$\begin{aligned} M_{hc}(D) :=& \{\eta \in M(D) : \forall x, y \in \eta,\ |x-y| > 1\} \\ =& \left\{ \eta \in M(D) : \int_D [\eta(B_1(x)) - 1] \eta(dx) = 0 \right\}, \end{aligned} \tag{2.1}$$

where $|x-y|$ denotes the Euclidean distance between x and y, and $B_1(x)$ denotes the ball of radius 1 around x. We note that $M_{hc}(D)$ is a measurable subset of $M(D)$. Also, set $\mathcal{M}_{hc}(D)$ to be restriction of $\mathcal{M}(D)$ to $M_{hc}(D)$.

Informally, a *Hard-core SwBD* process on D is a pure jump Markov process, $\{\eta_t\}_{t\geq 0}$, where jumps occur due to arrivals of particles. Arrival at a particle at location $x \in D$ occurs at rate $a(x, \eta_t)$ at time t, and the transitions in the state are governed by a transition kernel K. Here, $a : D \times M_{hc}(D) \to \mathbb{R}^+$ is a measurable functions, and $K : D \times M_{hc}(D) \times \mathcal{M}_{hc}(D) \to [0, 1]$ is a transition kernel. We assume that

$$K(x, \eta, \{\phi \in M_{hc}(D) : \phi \subset \eta \cup \{x\}\}) = 1, \tag{2.2}$$

so that the new state is formed by a subset of existing particles and the incoming particle. In this section, we also assume that $\sup_{\eta \in M_{hc}(D)} \int_D a(x, \eta)dx < \infty$.

More specifically, given that the state of the system at time t is $\eta_t \in M_{hc(D)}$, the probability of a particle arriving in any measurable subset $B \subset D$, in time $(t, t+\delta]$ is $\delta \int_B a(x, \eta_t)dx + o(\delta)$. When a particle arrives at location x at time t, then the state changes from η_{t-} to η, with η distributed according to $K(x, \eta_{t-}, \cdot)$.

To construct the process on a bounded set D, we first construct a Markov chain, $\{\Gamma_n\}_{n\in\mathbb{N}}$, such that the Γ_k is generated from Γ_{k-1} by sampling a point Y with density

$$\frac{a(\cdot, \Gamma_{k-1})}{\int_D a(x, \Gamma_{k-1})dx},$$

and then sampling Γ_k according to distribution $K(Y, \Gamma_{k-1}, \cdot)$. In addition, independent of everything else, let $\{\sigma_k\}_{k\geq 0}$ be independent exponentially distributed random variables with parameter 1. Finally, set

$$\tau_0 = 0, \qquad \tau_k = \sum_{j=0}^{k-1} \frac{\sigma_j}{\int_D a(x, \Gamma_j)dx},$$

and

$$\eta_t = \Gamma_k, \quad \tau_k \le t < \tau_{k+1}.$$

Instead of studying this general class of processes, in this chapter we focus on specific parameters, a and K, that we describe next. We assume that $a(x, \eta) = 1$ $\forall x,\ \eta$. We also set K so that a sample, $\Gamma \sim K(x, \eta, \cdot)$, is obtained from $\eta \cup \{x\}$ by following two steps

(i) For each $y \in \eta \cap B_1(x)$, among the pair $\{y, x\}$ either mark y, independently with probability ρ, or mark x otherwise. Here, $\rho \in [0, 1]$ is a parameter of the model.

(ii) Set Γ to be the set of unmarked points in $\eta \cup \{x\}$.

The motivation for using the above kernel is the following. If the current state of the process is $\eta \in M(D)$, and a point arrives at location x, then it *duels* with every point $y \in \eta \cap B_1(x)$. Each duel is independent of everything else and in each of these duels, either of the point is marked according a Bernoulli random variable. At the end of all the duels, all points marked are removed. Thus, for instance, the newly arriving point, x, is admitted to the system only if all points in $\eta \cap B_1(x)$ are removed, which occurs with probability $(1-\rho)^{|\eta \cap B_1(x)|}$.

One approach to give the definition of hard-core SwBD processes on unbounded domains is to extend the form of the generator from bounded to unbounded domains, and then study the function analytic properties of the generator to argue the existence of a probability semigroup. As an aside, we note that the generator of the process of interest on bounded domain D is

$$LF(\eta) = \int_D \rho^{\eta(B_1(x))}[F(\eta|_{B_1(x)^c} \cup \{x\}) - F(\eta|_{B_1(x)^c})] \\ + \sum_{\eta \subset \eta \cap B_1(x)} \rho^{|\eta|}(1-\rho)^{\eta(B_1(x))-|\eta|}[F(\eta \backslash \eta) - F(\eta)]\ dx, \tag{2.3}$$

for $F \in \mathrm{dom}(L)$. This approach would be along the lines of the work in [29], where the definition of a spatial birth-death process on an infinite domain is given by extending the definition from the bounded domain, i.e., it is set to have the same form.

In the next section we follow a different approach. We give an explicit construction of the process in a probability space. The algorithm is such that, if one restricts the domain to a bounded set and uses this algorithm for construction, one obtains the required hard-core SwBD process described above. The algorithm is similar to the *backward investigation and forward sweep* algorithm in [7], where the authors simulate an SBD process where deaths occur by random connections.

2.3.1 Construction of the process on $\mathbb{R}^d$

Let $\eta_0 \in M_{hc}(\mathbb{R}^d)$ be a random point process, that serves as the initial state for the hard-core SwBD process of interest. We assume that η_0 is stationary and ergodic. We further assume that the probability space is equipped with a marked point process $\hat{\Phi}$ on $\mathbb{R}^d \times \mathbb{R}^+$, with marks in $\{0,1\}^{\mathbb{N}}$. The point process Φ, which is the support of the marked point process $\hat{\Phi}$, is a homogeneous Poisson point process with intensity 1. The points of $\Phi \cup \eta_0$ are independently and identically marked, and further, for an arbitrary point of $x \in \Phi \cup \eta_0$, its mark $\{I_{x,y}\}_{y \in \Phi \cup \eta}$ are random variables that are independent and identically distributed Bernoulli random variables, with parameter ρ. An elementary application of the Kolmogorov's consistency theorem (see [17], Section 9.2) shows that one can define a probability space containing such a marked Poisson point processes and the initial point process η_0.

The point process $\hat{\Phi}$ serves as the arrival process. In particular, a particle $x \in \Phi$, with location $(p_x, b_x) \in \mathbb{R}^d \times \mathbb{R}$ is said to arrive at position p_x and at time b_x. The random variables $I_{x,y}$ represent the direction of killings: if x arrives after y then, y is marked in their duel if $I_{x,y} = 1$, otherwise if $I_{x,y} = 0$, then x is marked. The intuition for the algorithm to build the process η_t is that on a bounded measurable set D, we can use the following *forward sweep* algorithm:

- **Data:**
  1. The marked point process $\hat{\Phi}$.
  2. The realization of the initial condition, η_0.
  3. End time of the simulation $t > 0$.
- **Result:** The final state of the system at time t, η_t.

1. Set $t_{old} = 0$.
2. Set $t_{new} = \inf\{b_x : x \in \Phi, b_x > t_{old}\}$. If $t_{new} > t$, quit and return η_{t_old}.
3. Let the particle infimum correspond to $x = (p_x, b_x)$. Then

$$\eta_{t_{new}}|_{D \setminus B_1(p_x)} = \eta_{t_{old}}|_{D \setminus B_1(p_x)}, \tag{2.4}$$

and

$$\eta_{t_{new}}|_{D\cap B_1(p_x)} = \delta_{p_x} \prod_{y:p_y\in\eta_{t_{old}}|_{D\cap B_1(p_x)}} I_{x,y} + \sum_{y:p_y\in\eta_{t_{old}}|_{D\cap B_1(p_x)}} \delta_{p_y}(1 - I_{x,y}). \tag{2.5}$$

4. Set $t_{old} = t_{new}$. Go to Step 2.

It is easy to see that this forward sweeping algorithm on bounded domains indeed gives a construction of our process. We cannot use this algorithm to construct the restriction of the process on $\mathbb{R}^d$ to some bounded set D, since any particle arriving in D at time t, might depend on infinite number of arrivals that arrive outside D. Instead, in the following lemma, we describe a scheme to generate $\eta_t|_D$, at any time $t > 0$, by investigating backwards in time. We prove that this algorithm terminates when constructing η_t for any fixed time $t < \infty$.

Lemma 2.3.1. *Let $\hat{\Phi}$ be as described above and let η_0 be a stationary and ergodic hard-core point process. There is an $\epsilon > 0$ such that for any hard-core point process, η_0, there exists a hard-core SBD process η_t, $t \in [0, \epsilon]$, a.s., with arrivals from $\hat{\Phi}$ and the transitions described by eqs.* (2.4) *and* (2.5)*.*

Proof. We give an algorithm to construct the process for times $t \in [0, \epsilon]$, where $\epsilon > 0$ is fixed later. Let $\Phi_{[0,\epsilon]}$ denote the point process Φ restricted to the set $\mathbb{R}^d \times [0, \epsilon]$.

To construct η_t, it is enough to compute whether a particle $x \in \Phi_{[0,\epsilon]}$ is accepted when it arrives. We develop a directed dependency graph $G = (V, E)$, with vertex set $V = \Phi_{[0,\epsilon]}$ and $(x, y) \in E$ if and only if $p_y \in B_2(p_x)$ and $b_y < b_x$. So that $(x, y) \in E$ implies that, the acceptance of x at time b_x depends on the acceptance of

y at time b_y. Note that the radius of influence of a particle in $\Phi_{[0,\epsilon]}$ is set to 2 above, instead of 1, since an incoming particle can interact with a particle of η_0 within a distance 1 from it, which in turn may influence another particle of Φ within a distance at most 1 from it.

The projection of G (not considering directions of the edges) onto the spatial dimension is a Random geometric graph (see for e.g. [31] or [58]) on the projection of $\Phi_{[0,\epsilon]}$. That is, it is a Random geometric graph on a homogeneous Poisson point process with intensity $\lambda\epsilon$. From Theorem 2.6.1 of [31], if ϵ is small enough so that $\lambda\epsilon$ is less than a critical value, this graph does not percolate almost surely.

When every component of the graph is finite, the acceptance of any particle x in $\Phi_{[0,\epsilon]}$ can be calculated in finite time, by first focusing on the finite component of G to which x belongs, and calculate the status of x by running the forward sweep algorithm described before. Hence, there is an $\epsilon > 0$ so that, almost surely, for every point in $\Phi_{[0,\epsilon]}$, it may be evaluated whether the point is accepted when it arrives. This completes the proof. □

Note that in the proof above, ϵ is independent of the initial conditions. Hence, using the above lemma, we can construct the required process successively on time intervals $[n\epsilon, (n+1)\epsilon]$, $n \in \mathbb{N}$, starting with any initial condition. This result is summarized in the following corollary.

Corollary 2.3.2. *Under the hypothesis of Lemma 2.3.1, there exists a hard-core SwBD process η_t, $t \in [0, \infty)$, with arrivals in Φ and local interactions given by* (2.4) *and* (2.5)*. Further, for any time* $t > 0$*,* η_t *is spatially stationary and ergodic if* η_0 *is spatially stationary and ergodic.*

Proof. The above discussion gives the construction of the process η_t for any finite time interval $t \in [0, T]$.

For any $t > 0$, η_t is spatially stationary and ergodic since it is a translation invariant thinning of the spatially stationary and ergodic projection of the process $\Phi_{[0,t]} \cup \eta_0$. □

2.4 Time Stationarity and Ergodicity

In this section we show, under certain assumptions, the existence of a stationary regime for the process constructed in the previous section. While the method is non-constructive, it is inspired from Perfect-simulation or Coupling from the past techniques. A similar technique was used in [7] for proving the existence of a stationary regime in case of their *death by random connection* model. We need to show the exponential decay of discrepancies between two coupled processes starting from distinct initial conditions. This exponential decay in turn also proves the uniqueness of the stationary distribution. This argument is spelled out in Section 2.4.3, where it is also shown that the distribution of the state of Markov process at time t converges to that of the stationary state as $t \to \infty$.

The key ideas of the Coupling from the past technique below are the following: We run the process using a fixed temporally stationary ergodic driving process from time $-T$ until time t. We call this process η_t^T. We then show that the limit, as

$T \to \infty$, of the random process, η_t^T, converges almost surely. This is done by showing that any two process started at time 0 couple in a time that has finite expectation. The ergodic theorem can then be used to claim that the limiting process exists almost surely.

Accordingly, in the following section we define a coupling of two hard-core SwBD processes and produce sufficient conditions for exponential decay of density of discrepancies between them. Later, in Section 2.4.2, we construct the stationary regime using the coupling from the past argument.

2.4.1 Coupling of Two Processes; Density of "Special" Points

In this section, we consider processes $\{\eta^1\}_{t\geq 0}$ and $\{\eta^2\}_{t\geq 0}$ driven by the same marked Poisson point process, $\hat{\Phi}$, with η_0^1 and η_0^2 being a spatially stationary and ergodic hard-core point process. At any time $t > 0$, there are some points that are present in both processes. These points are called *regular* points, and they form a stationary point process we denote R_t. The remaining points are present in one of the processes and absent in the other. These points are referred to as *special* points and the symbol S_t is used to denote the point process formed by these. In particular, the points present in η_t^1 and absent in η_t^2 are called antizombies, denoted by A_t; and those points alive in η_t^2 and dead in η_t^1 are called zombies, denoted by Z_t. Thus,

$$R_t = \eta_t^1 \cap \eta_t^2,\ S_t = \eta_t^1 \triangle \eta_t^2,\ A_t = \eta_t^1 \backslash \eta_t^2,\ \text{and}\ Z_t = \eta_t^2 \backslash \eta_t^1.$$

We need the following notation. If X is a stationary point process in $\mathbb{R}^d$, then we denote β_X to be the intensity of the point process X, i.e., $\beta_X = \mathrm{E}\, X([0,1]^d)$. Thus,

β_{R_t} and β_{S_t} denotes the intensity of the point processes R_t and S_t respectively.

We now prove the following theorem concerning the exponential decay of the intensity of the special points, S_t. Exponential decay will be used to prove the existence of a stationary regime. This result can be interpreted as saying that the information about the initial state is erased sufficiently quickly by the incoming points.

Theorem 2.4.1. *If $\rho > 1/2$, then there exists constants, $\alpha, c > 0$ such that $\beta_{S_t} < \alpha e^{-ct}$, for all $t > 0$.*

Proof. Suppose $z \in S_t$ is a special point. Given the realization of $\hat{\Phi}$, we build an interaction graph G on the points $S_t \cup \Phi_{(t,\infty)}$, with directed edges (x, y) if and only if $b_x < b_y$, $p_x \in S_{b_y-} \cap B_1(p_y)$ and $I_{y,x} = 0$. Let G_z be the subgraph of G containing z, called the *family* of z. Let $M_{z,t,t+\delta}$ denote the elements of G_z alive at time $t + \delta$, with $m_{z,t,t+\delta} = |M_{z,t,t+\delta}|$. These are the only points on Φ that interact with z and that could belong $S_{t+\delta}$

Using Mass transport principle (see Theorem 4.1 in [49]), with unit mass outgoing from each point in S_t to all point in $S_{t+\delta}$ in its family, we get the following bound:

$$\beta_{S_{t+\delta}} \leq \beta_{S_t} \, \mathrm{E}^0_{S_t} \, m_{0,t,t+\delta}. \tag{2.6}$$

We do not get equality in the above expression because a special point in $S_{t+\delta}$ can belong to more than one family.

We have, by superposition principle,

$$\beta_{S_t} \mathrm{E}^0_{S_t} m_{0,t,t+\delta} = \beta_{Z_t} \mathrm{E}^0_{Z_t} m_{0,t,t+\delta} + \beta_{A_t} \mathrm{E}^0_{A_t} m_{0,t,t+\delta}. \tag{2.7}$$

Let $i_z = |\Phi_{(t,t+\delta)} \cap [B_1(z) \times (t, t+\delta)]|$, $j_z = |\Phi_{(t,t+\delta)} \cap [B_2(z) \times (t, t+\delta)]|$ and D_z be the event that the interaction graph of z between time t and $t+\delta$ contains at least two points. Let ν_1 be the volume of the ball of radius 1. By spatial stationarity we can work with the Palm expectation, assuming that the location z is at 0.

$$\begin{aligned}\mathrm{E}^0_{Z_t} m_{0,t,t+\delta} \leq & \mathrm{P}^0_{Z_t}(i_0 = 0) + \mathrm{P}^0_{Z_t}(i_0 = 1) \mathrm{E}^0_{Z_t}[m_{0,t,t+\delta} | i_0 = 1, D_0^c] \\ & + \mathrm{P}^0_{Z_t}(D_0) \mathrm{E}^0_{Z_t}[\mathbb{1}_{D_0} m_{0,t,+\delta} | D_0].\end{aligned}$$

We note that the last term in the above expression is $o(\delta)$. Indeed, $m_{0,t,t+\delta}$ can be stochastically bounded by a pure birth process with birth rates $\lambda_k = k\nu_1$, $k \geq 1$, as each successive member of a family increases the coverage area of the family by at most ν_1. Then using the explicit probability distributions for this pure-birth process it is easy to conclude this result (see [71] Chapter 5).

Hence,

$$\mathrm{E}^0_{Z_t} m_{0,t,t+\delta} \leq e^{-\nu_1\lambda\delta} + \nu_1\lambda\delta e^{-\nu_1\lambda\delta} \mathrm{E}^0_{Z_t}[m_{0,t,t+\delta} | i_0 = 1, D_0^c] + o(\delta).$$

Define $\mathfrak{R}_t = \cup_{x \in R_t} B_1(x)$, $\mathfrak{Z}_t = \cup_{x \in Z_t} B_1(x)$ and $\mathfrak{A}_t = \cup_{x \in A_t} B_1(x)$. We have the following possible disjoint events within $\{i_0 = 1\} \cap D_0^c$ depending on where the particle arrives:

E_1: If the particle arrives in $B_1(0) \cap (\mathfrak{R}_t \cup \mathfrak{A}_t)^c$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } 1-\rho \\ 0 & \text{w.p. } \rho \end{cases}.$$

Hence, $\mathrm{E}^0_{Z_t}[m_{0,t,t+\delta}|i_0 = 1, D_0^c, E_1] = 2(1-\rho)$.

E_2: If the particle arrives in $B_1(0) \cap \mathfrak{R}_t \cap \mathfrak{A}_t^c$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } \rho^k(1-\rho) \\ 1 & \text{w.p. } (1-\rho)(1-\rho^k) \\ 0 & \text{w.p. } \rho \end{cases},$$

where k is the number of regular points that interact with the incoming point.

Hence,

$$\mathrm{E}^0_{Z_t}[m_{0,t,t+\delta}|i_0 = 1, D_0^c, E_2] \le (1-\rho^2).$$

E_3: If the particle arrives in $B_1(0) \cap \mathfrak{A}_t \cap \mathfrak{R}_t^c$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } \rho^k(1-\rho) \\ 1 & \text{w.p. } (1-\rho)(1-\rho^k) \\ 0 & \text{w.p. } \rho \end{cases},$$

where k is the number of antizombies that interact with the incoming point.

Hence,

$$\mathrm{E}^0_{Z_t}[m_{0,t,t+\delta}|i_0, D_0^c, E_3] \le (1-\rho^2).$$

E_4: If the particle arrives in $B_1(0) \cap \mathfrak{R}_t \cap \mathfrak{A}_t$, then

$$m_{0,t,t+\delta} = \begin{cases} 2 & \text{w.p. } \rho^k(1-\rho) \\ 1 & \text{w.p. } (1-\rho)(1-\rho^k) \\ 0 & \text{w.p. } \rho \end{cases},$$

where k is the number of regular and antizombies that the point interacts with.

Therefore,

$$\mathrm{E}^0_{Z_t}[m_{0,t,t+\delta}|i_0 = 1, D_0^c, E_4] \le (1-\rho^2).$$

Consequently, taking all these 4 cases into account we obtain,

$$
\begin{aligned}
&\mathrm{E}^0_{Z_t}[m_{0,t,t+\delta}|i_0 = 1, D_0^c] \\
&\le 1 - \rho^2 + \frac{1}{\nu_1}\mathrm{E}^0_{Z_t}\left((2(1-\rho) - (1-\rho^2))\ell(B_1(0) \cap (\mathfrak{R}_t \cup \mathfrak{U}_t)^c)\right) \\
&= 1 - \rho^2 + \left[\frac{(1-\rho)^2}{\nu_1}\mathrm{E}^0_{Z_t}\,\ell(B_1(0) \cap (\mathfrak{R}_t \cup \mathfrak{A}_t)^c)\right] \\
&= 1 + \left[1 - 2\rho - \frac{(1-\rho)^2}{\nu_1}\mathrm{E}^0_{Z_t}\,\ell(B_1(0) \cap (\mathfrak{R}_t \cup \mathfrak{U}_t))\right] \\
&\le 1 + \left[1 - 2\rho - \frac{(1-\rho)^2}{\nu_1}\mathrm{E}^0_{Z_t}\,\ell(B_1(0) \cap \mathfrak{R}_t)\right].
\end{aligned}
\tag{2.8}
$$

Here, $\ell(\cdot)$ denote the Lebesgue measure on $\mathbb{R}^d$. From eqs. (2.6)-(2.8),

$$
\begin{aligned}
\beta_{S_{t+\delta}} - \beta_{S_t} \le \beta_{S_t}\Bigg[&e^{-\nu_1\lambda\delta}(1+\nu_1\lambda\delta) - 1 \\
&+\nu_1\lambda\delta e^{-\nu_1\lambda\delta}\left(1 - 2\rho - \frac{1}{\nu_1}(1-\rho)^2\,\mathrm{E}^0_{S_t}\,\ell(B_1(0)\cap\mathfrak{R}_t)\right) + o(\delta)\Bigg].
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\frac{1}{\beta_{S_t}}\frac{d\beta_{S_t}}{dt} &= \frac{1}{\beta_{S_t}}\limsup_{\delta\to 0}\frac{\beta_{S_{t+\delta}} - \beta_{S_t}}{\delta} \\
&\le \nu_1\lambda\left((1-2\rho) - \frac{1}{\nu_1}(1-\rho)^2\,\mathrm{E}^0_{S_t}\,\ell(B_1(0)\cap\mathfrak{R}_t)\right) \qquad (2.9)\\
&\le \nu_1\lambda(1-2\rho). \qquad (2.10)
\end{aligned}
$$

Hence, if $\rho > \frac{1}{2}$, then we see that β_{S_t} decreases exponentially to zero, i.e., there exists $c > 0$, such that $\beta_{S_t} \le \beta_{S_0}e^{-ct}$. □

In the next result, we utilize the geometry of the interactions to gain more from the inequality (2.9). In particular, we prove that the intensity of special points decays exponentially quickly even for ρ slightly less than 1/2. Another interesting feature of the proof of next result is the use of differential equations second-order moments of various point processes. Note that, in the previous proof, we have only written differential equations of the first-order moment measures. We will need to define the well-known geometric constant called the kissing number, $\kappa = \kappa(d)$. This is defined as the maximum number of non-overlapping unit spheres that can be arranged such that they each touch another fixed unit sphere. It depends on the dimension, d. For example, $\kappa(1) = 2$ and $\kappa(2) = 6$.

Theorem 2.4.2. *If*

$$\left((1-2\rho) - \frac{\rho^{\kappa}(1-\rho)^2((\frac{3}{2})^d - 1)}{4^d(\kappa-1)(1+\rho-\rho^2)}\right) < 0,$$

then there exist constants $\alpha, c > 0$ *such that for all* $t > 0$, $\beta_{S_t} \le \alpha e^{-ct}$.

Proof. We now note that

$$\ell(B_1(0) \cap \mathfrak{R}_t) \ge \frac{\nu_1}{4^d} \mathbb{1}(R_t(B_{\frac{3}{2}}(0)) > 0)) \ge \frac{\nu_1}{4^d(\kappa-1)} R_t(B_{\frac{3}{2}}(0)), \tag{2.11}$$

where the first inequality, we ignore regular points with distance greater than 3/2, and then note that $B_1(0) \cap B_1(x)$ contains a ball of radius $\frac{1}{4}$ if $|x| \le \frac{3}{2}$. The second inequality is true since $R_t(B_{\frac{3}{2}}(0))$ takes only values $0, 1, \dots, \kappa - 1$.

Hence, to bound $\mathrm{E}^0_{S_t} \ell(B_1(0) \cap \mathfrak{R}_t)$ from below we calculate bounds on the derivative of $\beta_{S_t} \mathrm{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0))$. If $C \subset \mathbb{R}^d$ is a set of measure 1, then

$$\beta_{S_t} \mathrm{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)) = \mathrm{E}\left[\sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B_{\frac{3}{2}}(x)} 1\right].$$

The derivative of the above expression depends on rates of increase and decrease of both regular and special points. We now give a lower bound on the derivative by accounting for various types of interactions.

We first consider the killings (rate of decrease).

- For each point $x \in S_t \cap C$, a new point could arrive from the Poisson rain and kill x with probability ρ. This type of interaction results in a rate equal to

$$-\nu_1 \lambda \rho \, \mathrm{E}\left[\sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B_{\frac{3}{2}}(x)} 1\right] = -\nu_1 \lambda \rho \beta_{S_t} \, \mathrm{E}^0_{S_t} \, R_t(B_{\frac{3}{2}}(0)).$$

- For each point $x \in S_t \cap C$ and $y \in R_t \cap B_{\frac{3}{2}}(x)$, a new point could arrive in $B_1(x) \cap \mathfrak{R}_t$, the point x survives, but y is killed. This results in a rate equal to

$$\begin{aligned}
&- \lambda \rho (1-\rho) \, \mathrm{E}\left[\sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B_{\frac{3}{2}}(x)} \ell(B_1(x) \cap B_1(y))\right] \\
&\geq -\lambda \rho (1-\rho) \, \mathrm{E}\left[\sum_{x \in S_t \cap C} \sum_{y \in R_t \cap B_{\frac{3}{2}}(x)} \nu_1\right] \\
&= -\nu_1 \lambda \rho (1-\rho) \beta_{S_t} \, \mathrm{E}^0_{S_t} \, R_t(B_{\frac{3}{2}}(0)).
\end{aligned}$$

- For each point $x \in S_t \cap C$ and $y \in R_t \cap B_{\frac{3}{2}}(x)$, a new point could arrive in the region $B_1(y) \cap B_1(x)^c$ and kill y. This results in a rate of change equal to

$$\begin{aligned}
-\lambda \rho \, \mathrm{E}\left[\sum_{S_t \cap C} \sum_{y \in R_t \cap B_{\frac{3}{2}}(x)} \ell(B_1(y) \cap B_1(x)^c)\right] &\geq - \lambda \rho \, \mathrm{E}\left[\sum_{S_t \cap C} \sum_{y \in R_t \cap B_{\frac{3}{2}}(x)} \nu_1\right] \\
&= - \nu_1 \lambda \rho \beta_{S_t} \, \mathrm{E}^0_{S_t} \, R_t(B_{\frac{3}{2}}(0)).
\end{aligned}$$

We now consider the rate of increase:

- For each $x \in S_t \cap C$, a new point could arrive at $B_{\frac{3}{2}}(x) \backslash B_1(x)$ and compete with other points. It survives and becomes a regular with a probability at least ρ^κ. This type of interaction results in a rate at least equal to $\rho^\kappa \left(\left(\frac{3}{2}\right)^d - 1 \right) \nu_1 \lambda \beta_{S_t}$. Let $c_0 := \rho^\kappa \left(\left(\frac{3}{2}\right)^d - 1 \right)$.

- We ignore the rate at which new special points are created.

Consequently,

$$\frac{d}{dt} \beta_{S_t} \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)) \geq c_0 \nu_1 \lambda \beta_{S_t} - \rho(3-\rho) \nu_1 \lambda \beta_{S_t} \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)).$$

Using the product rule of differentiation and (2.10), we obtain:

$$\begin{aligned} \frac{d}{dt} \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)) &\geq c_0 \nu_1 \lambda - \nu_1 \lambda \rho (3-\rho) \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)) - \frac{1}{\beta_{S_t}} \frac{d\beta_{S_t}}{dt} \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)) \\ &\geq c_0 \nu_1 \lambda - \left(1 + \rho - \rho^2\right) \nu_1 \lambda \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)). \end{aligned}$$

This shows that

$$\liminf_{t \to \infty} \operatorname{E}^0_{S_t} R_t(B_{\frac{3}{2}}(0)) \geq \frac{c_0}{1 + \rho - \rho^2} = \frac{\rho^\kappa \left(\left(\frac{3}{2}\right)^d - 1 \right)}{1 + \rho - \rho^2}.$$

From eqs. (2.11) and (2.9), it then follows that

$$\limsup_{t \to \infty} \frac{1}{\beta_{S_t}} \frac{d\beta_{S_t}}{dt} \leq \nu_1 \lambda \left((1 - 2\rho) - \frac{\rho^\kappa (1-\rho)^2 ((\frac{3}{2})^d - 1)}{4^d (\kappa - 1)(1 + \rho - \rho^2)} \right).$$

Thus, using Gronwall's theorem we arrive at the desired result. $\square$

The last result also gives a better range of ρ for which coupling from the past argument can be performed. For $d = 1$, the above sufficient condition is satisfied when $\rho > 0.497$. However, in higher dimensions, since the kissing number increases exponentially with dimension, this bound may not be close to optimal. We expect that a more detailed analysis may provide a better lower bound.

2.4.2 Coupling from the Past

In this section we construct a stationary regime for the hard-core SwBD process using the method of coupling from the past. Consider a doubly infinite marked Poisson point process $\hat{\Phi}$ on $\mathbb{R}^d \times (-\infty, \infty)$ with mean measure $\lambda \ell(dx) \times dt$, with i.i.d marks as described earlier. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a group of time-shift operators under which the point process Φ is ergodic. Let η_t be the process starting at time 0 with empty initial condition. Now, consider the sequence of processes $\{\eta_t^T, t > -T\}_{T\in\mathbb{N}}$, obtained with empty initial condition from time $-T$ by using Φ, for $T \in \mathbb{N}$. We have $\eta_t^T = \eta_{t+T} \circ \theta_{-T}$.

The processes η_t^1 and η_t^0 are driven by the same Poisson process beyond time 0. Treating the points in η_0^1 as initial conditions for the augmented process (Zombies), if the sufficient conditions of Theorem 2.4.2, hold then the density of special points in the coupling of η_t^1 and η_t^0 goes to zero exponentially quickly. The following statement shows that the two processes coincide on a compact space in finite time with finite expectation. For a compact set $C \subset \mathbb{R}^d$, let

$$\tau(C) := \inf\{t > 0 : \eta_s^1|_C = \eta_s^0|_C,\ s \geq t\}.$$

Note that $\tau(C)$ is not a stopping time.

In the following lemma, recall that S_t is the set of discrepancies, $\eta_t^0 \triangle \eta_t^1$.

Lemma 2.4.3. *If for some $c > 0$, $\beta_{S_t} \leq \beta_{S_0} e^{-ct}$, then the minimum time $\tau(C)$ beyond which η_t^0 and η_t^1 coincide on the set C has finite expectation.*

Proof. We view $S_t(C)$, $t > 0$, as a simple birth-death process. Let $S_t(C) = S_0(C) + S^+[0,t] - S^-[0,t]$, where S^+ and S^- are point processes on $\mathbb{R}^+$ with the following properties:

1. S^+ is a simple counting process, with a jump indicating the arrival of a new special point in C.

2. S^- is a counting process, with a jump indicating the departure of corresponding number of special points from C.

Since special points result from interaction of arriving points with existing special points, the rate of increase in S^+ is bounded above by $S_t(C \oplus B_1(0)) \times \lambda\nu_1$. Hence,

$$\begin{aligned} \mathrm{E}\, S^+[0,\infty) &\leq \lambda\nu_1 \int_0^\infty \mathrm{E}\, S_t(C \oplus B_1(0)) dt \\ &= \lambda\nu_1 \ell(C \oplus B_1(0)) \int_0^\infty \beta_{S_t} dt < \infty. \end{aligned}$$

This also shows that $S^+[0,\infty)$ and $S^-[0,\infty)$ exist and are finite a.s. Thus,

$$\lim_{t\to\infty} S_t(C)$$

also exists and is finite. From the fact that

$$\lim_{t\to\infty} \mathrm{E}\, S_t(C) = \lim_{t\to\infty} \beta_{S_t} \ell(C) = 0,$$

by dominated convergence theorem, we have $\mathrm{E}\lim_{t\to\infty} S_t(C) = 0$. Thus,

$$\lim_{t\to\infty} S_t(C) = 0, \text{ a.s.}$$

This also shows that $\tau(C) < \infty$ a.s.

Let S be the random measure on $\mathbb{R}^+$, with $S[0,t] = S_t(C)$ for all $t \geq 0$. We have

$$\begin{aligned}
\mathrm{E}\,\tau(C) &\leq \mathrm{E}\int_0^\infty tS^-(dt) \\
&= \mathrm{E}\int_0^\infty tS^+(dt) - \mathrm{E}\int_0^\infty tS(dt) \\
&\leq \lambda\nu_1\ell(C \oplus B_1(0))\int_0^\infty t\beta_{S_t}dt + \mathrm{E}\int_0^\infty S[0,t]dt \\
&= \lambda\nu_1\ell(C \oplus B_1(0))\int_0^\infty t\beta_{S_t}dt + \ell(C)\int_0^\infty \beta_{S_t}dt < \infty.
\end{aligned}$$

□

Now, let V_y^T denotes the time at which the executions of processes η_t^T and η_t^{T+1} coincide in $B_1(y)$, i.e., $V_y^T = \tau(B_1(y)) \circ \theta_{-T} - T$. Then, we have:

$$V_y^T + T = \tau(B_1(y)) \circ \theta_{-T} = V_y^0 \circ \theta_{-T}. \tag{2.12}$$

Consequently, we have the following lemma

Lemma 2.4.4. *If* $\mathrm{E}\,\tau(B_1(y)) < \infty$, *then*

$$\lim_{T\to\infty} V_y^T = -\infty. \tag{2.13}$$

Proof. By (2.12), $V_y^T + T$ is a stationary and ergodic sequence. Hence, by Birkhoff's pointwise ergodic theorem,

$$\lim_{T\to\infty} \sum_{i=0}^{T} \frac{|V_y^i + i|}{T} = \mathrm{E}\,\tau(B_1(y)) < \infty, \text{ a.s.}$$

Therefore the last term in the summation, $\frac{V_y^T + T}{T} \to 0$ a.s., as $T \to \infty$. This implies the desired result that

$$\lim_{T\to\infty} V_y^T = -\infty, \text{ a.s.}$$

□

Thus, for every realization of Φ, any compact set C and $t \in \mathbb{R}$, there exists a $k \in \mathbb{N}$ such that for all $T > k$, $\tau(C) \circ \theta_{-T} - T < t$. That is, for $T > k$, the execution of all the processes starting at $-T$ coincide at time t on the compact set C. The limit

$$\Upsilon_t := \lim_{T\to\infty} \eta_{t+T} \circ \theta_{-T} \tag{2.14}$$

may now be defined as the weak limit of restrictions to compact sets, $\Upsilon_t|_C = \lim_{T\to\infty} \eta_{t+T} \circ \theta_{-T}|_C$. Further,

$$\begin{aligned}
\Upsilon_t \circ \theta_1 &= \lim_{T\to\infty} \eta_{t+T} \circ \theta_{-T+1} \\
&= \lim_{T\to\infty} \eta_{t+1+T-1} \circ \theta_{-T+1} \\
&= \Upsilon_{t+1}.
\end{aligned}$$

So, the process Υ is $\{\theta_n\}_{n\in\mathbb{Z}}$ compatible. In fact, Υ is also $\{\theta_s\}_{s\in\mathbb{R}}$ compatible and temporally ergodic, since it is a factor of the driving process Φ. Thus, Υ is the stationary regime for this process.

2.4.3 Convergence in Distribution

Let $\{\eta^t\}_{t\geq 0}$ be a Hard-core SwBD process driven by a homogeneous Poisson point process $\hat{\Phi}$ on $\mathbb{R}^d \times \mathbb{R}^+$, as described in Section 2.3, with ergodic initial conditions. Let Υ_0 be stationary regime of the process at time zero. Consider the process $\hat{\eta}_t$ with initial condition $\hat{\eta}_0 = \Upsilon_0$ and being driven by the point process $\hat{\Phi}$. Note that $\hat{\eta}_t \stackrel{d}{=} \Upsilon_t \stackrel{d}{=} \Upsilon_0$. If the conditions of Theorem 2.4.2 are satisfied then we can conclude that the density of the discrepancies between the two processes vanishes exponentially to zero. This gives the following quantitative estimate on the difference of the Laplace functional $\mathcal{L}_t$ and $\hat{\mathcal{L}}_t$ of η_t and $\hat{\eta}_t$ respectively.

Lemma 2.4.5. *Let $S_t = \eta_t \triangle \hat{\eta}_t$. If there exists $c > 0$ such that $\beta_{S_t} \leq \beta_{S_0} e^{-ct}$, then for any $f \in \mathcal{G}_b(\mathbb{R}^d) \cap \mathcal{G}_+(\mathbb{R}^d)$, we have*

$$|\mathcal{L}_t(f) - \hat{\mathcal{L}}_t(f)| \leq \|f\|_{L^1} \beta_{S_0} e^{-ct} \tag{2.15}$$

Proof. Let $f \in BM_+(\mathbb{R}^d)$. Then,

$$\begin{aligned}
\mathcal{L}_t(f) - \hat{\mathcal{L}}_t(f) &= \mathrm{E}\left[e^{-\int f(x)\eta_t(dx)}\left(1 - \prod_{x\in\hat{\eta}_t\setminus\eta_t} e^{-f(x)} \prod_{x\in\eta_t\setminus\hat{\eta}_t} e^{f(x)}\right)\right] \\
&\leq \mathrm{E}\left[e^{-\int f(x)\eta_t(dx)}\left(\int f(x)\hat{\eta}_t\setminus\eta_t(dx) - \int f(x)\eta_t\setminus\hat{\eta}_t(dx)\right)\right].
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathcal{L}_t(f) - \hat{\mathcal{L}}_t(f) &= \mathrm{E}\left[e^{-\int f(x)\hat{\eta}_t(dx)}\left(\prod_{x\in\hat{\eta}_t\setminus\eta_t} e^{f(x)} \prod_{x\in\eta_t\setminus\hat{\eta}_t} e^{-f(x)} - 1\right)\right] \\
&\geq \mathrm{E}\left[e^{-\int f(x)\hat{\eta}_t(dx)}\left(\int f(x)\hat{\eta}_t\setminus\eta_t(dx) - \int f(x)\eta_t\setminus\hat{\eta}_t(dx)\right)\right].
\end{aligned}$$

Hence,

$$
\begin{aligned}
&|\mathcal{L}_t(f) - \hat{\mathcal{L}}_t(f)| \\
&\leq \mathrm{E}\left[\max\left\{e^{-\int f(x)\hat{\eta}_t(dx)}, e^{-\int f(x)\eta_t(dx)}\right\}\left|\int f(x)\hat{\eta}_t\backslash\eta_t(dx) - \int f(x)\eta_t\backslash\hat{\eta}_t(dx)\right|\right] \\
&\leq \mathrm{E}\left|\int f(x)\hat{\eta}_t\backslash\eta_t(dx) - \int f(x)\eta_t\backslash\hat{\eta}_t(dx)\right| \\
&\leq \mathrm{E}\int f(x)S_t(dx) \\
&= \beta_{S_t}\|f\|_{L^1} \\
&\leq \beta_{S_0}\|f\|_{L^1}e^{-ct}.
\end{aligned}
$$

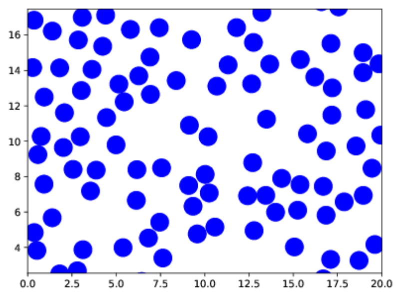
□

Since point-wise convergence of Laplace functional also implies convergence in distribution, we can conclude that η_t converges weakly to Υ_0 as $t \to \infty$.

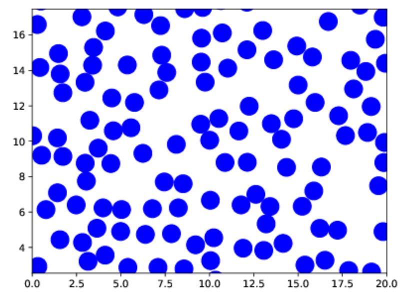
2.5 Concluding Remarks and Future Work

In this chapter, we focused on the Hard-core SwBD process on an infinite domain where the interactions are pairwise. It was shown that under the conditions of Theorem 2.4.2, a stationary regime exists.

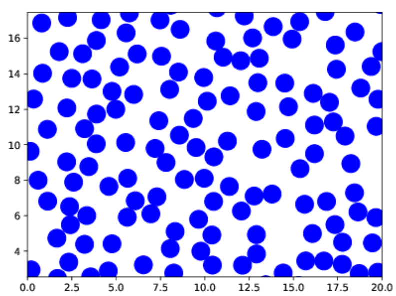
It remains to be seen if exponential convergence as above can be shown for all value of $\rho > 0$. Differential equations of higher order moment measures might be necessary for controlling the decay in density of special points in this case. The RSA scheme ($\rho = 0$), for example is a pathwise monotonic process, and multitude of stationary distributions that heavily dependent on the initial arrivals. As in lattice



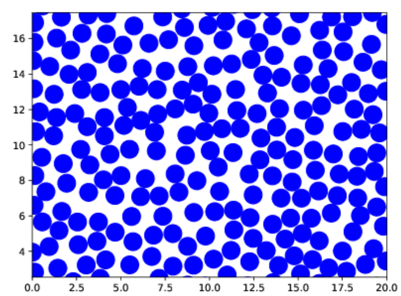
(a) $\rho = 0.8$, PF$= 0.247$



(b) $\rho = 0.5$, PF$= 0.291$



(c) $\rho = 0.2$, PF$= 0.327$



(d) $\rho = 0$, PF$= 0.542$

Figure 2.1: Samples from the stationary state on a finite window. The packing fraction is observed to increase as $\rho \to 0$.

models of interacting particle systems, we suspect that there exists a critical value, $\rho_c > 0$, such that for all $\rho < \rho_c$ the stationary distribution is non-unique, while for $\rho > \rho_c$ the stationary distribution is unique.

Further, it would also be useful to obtain quantitative bounds on the packing efficiencies of the hard-core SwBD processes considered here, in their stationary regimes. Figure 2.1 presents simulation results for certain values of ρ on a bounded region of the plane. For $\rho = 0.5$, these simulations indicate that the packing efficiency is close to 0.29. While this is much less than the RSA scheme, whose packing fraction in the jamming limit is predicted to be around 0.54, the existence of a unique stationary regime means that the process meets the primary requirement for fairness of a resource allocation scheme. The dependence of the packing efficiency on the parameter ρ is also unclear. Further, packing efficiencies of other hard-core point processes need to be compared with the hard-core SwBD processes. In particular, one class of processes where the hard-core structure shows up are the Matérn type-I and type-II processes [57]. These processes can be considered as a dependent thinning of Poisson point processes, based on a retention rule. The packing efficiencies of these Matérn processes are known, and they form an interesting class for comparison of packing efficiencies.

Chapter 3

The Gibbs Hard-Core Process

3.1 Introduction

In this chapter, we study a classical spatial birth-death process whose state space satisfies the hard-core condition, that we refer to as the *Gibbs hard-core* process. This process also appears in literature with the name Glauber dynamics of the Hard-core Gibbs measure in the continuum.

Let us consider a domain $D \subset \mathbb{R}^d$, for some $d \geq 1$. A Gibbs hard-core process on D with *activity* $\lambda > 0$, with hard-core distance 1, is a spatial birth-death process (see Section 1.1.2) on domain D with birth and death rate functions

$$b(x, \eta) = \begin{cases} \lambda & \text{if } \eta(B_1(x)) = 0, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad d(x, \eta) = 1 \tag{3.1}$$

where $B_1(x)$, is the ball of radius 1 around the point x. Thus, a new particle cannot appear at location x, if there is another particle already in the system within a distance one from it. All accepted particles disappear at rate 1.

When D is a bounded domain, this process can be constructed as a Markov jump process. However, when D is unbounded the question of existence or the construction of this process is non-trivial, since there is no notion of a first arrival in an unbounded domain of infinite measure. Beyond the problem of existence or construction of this process, we are interested in understanding its steady state behavior of this Markov process. When D is bounded, we have a complete characterization of the stationary regime of the Gibbs hard-core process in terms of the Papangelou conditional intensity (see Section 1.1.1 for a definition):

$$\lambda^*(x,\eta) = \frac{b(x,\eta)}{d(x,\eta)} = \lambda \mathbb{1}(\eta(B_1(x)) = 0). \tag{3.2}$$

In this setting, we are interested in the rate of convergence of the process to its stationary regime. When D is infinite, the existence of a stationary measure, or a measure with Papangelou conditional intensity in (3.2) is non-trivial.

In this chapter, we will employ the techniques from Chapter 2 to study this process. In particular, we will give a sufficient an bound on λ, below which we are able to construct a stationary regime for the this process in the infinite Euclidean domain. We then survey techniques and results along the same line that are available in literature. The overall upshot in this presentation is that stationarity can be proved for small values of activity parameter λ, and the goal is to improve these upper bounds.

3.1.1 Related work

The dynamics of the Gibbs hard-core model considered above shows up under many names in the literature. Informally, this point process can be simulated on a bounded domain D as follows. Particles arrive according to a Poisson point process with intensity λ on D. Once a particle arrives, it is accepted into the system if there are no particles in the system within a distance one from it. Once accepted, a particle stays in the systems for an exponential amount of time before departing. The state of the system at any time is the set of locations of the particles present in the system.

In [26, 33], a related model called the Continuous Loss Network model was studied. This process was studied in the one-dimension, where it serves as a model for telephonic network line. In [34] the authors give a *coupling from the past* scheme to obtain a steady state. This scheme works when λ is small enough. The coupling from the past scheme also shows the uniqueness of the stationary state for this range of values of λ. However, as λ increasing it is expected that there will be multiple steady states, and there is a critical value λ_c, beyond which this happens. Indeed, the Random Sequential Adsorption (RSA) model [65, 62] trivially has infinitely many steady states, and can be considered as a limiting process obtained as $\lambda \to \infty$. In the RSA model on $\mathbb{R}^d$, disks of radius $\frac{1}{2}$ arrive in the system, and an arriving disk is accepted if it does not overlap with any disk already accepted into the system. An accepted disk never leaves the system. Thus, the RSA model can be seen as an SBD process with pure births. It is an appropriate model for modeling many physical, chemical and biological processes [25], particularly when irreversible deposition or reactions are involved.

On bounded domains, the Gibbs hard-core process is reversible, and the stationary distribution can be shown to be the Hard-core Gibbs distribution, specifically with Papangelou conditional intensity given in (3.2). On unbounded domains, the existence of a measure with Papangelou conditional intensity (3.2) is non-trivial.

A related class of processes is the Glauber dynamics of the hard-core model on a graph G. The hard-core model graphs was studied in statistical physics as a model of a lattice gas (see [68]), and in operations research where it was used as a model for a communication network (see [44]). Here, the graph G can be either finite or infinite. Particles arrive according to a Poisson rain process on the vertices of G with intensity λ, and a new particle is accepted if there is no particle at the vertex, or at any vertex adjacent to it, that is already in the system. When the graph G is finite, the stationary distribution is the Hard-core Gibbs distribution, $\mu_{G,\lambda}$, where the probability of observing an independent set I, $\mu_{G,\lambda}(I)$, is proportional to $\lambda^{|I|}$, where $|I|$ denotes the number of vertices in I.

It is a common practice in statistical mechanics to consider a model in infinite lattices ($\mathbb{Z}^d$ for example) for studying extrinsic/macroscopic properties of extremely large systems. For infinite graphs, the Gibbs hard-core distribution on graphs is defined as a suitable weak limit of a consistent family of conditional distributions (see [19]). Every Gibbs distribution on a given infinite graph is a stationary distribution of the *Glauber dynamics* on the infinite graph. As in the classical Ising model, it is conjectured that the infinite system undergoes a phase transition as the parameter λ is increased. For d-regular trees, in [44] it was shown that this critical value is $\lambda_c = d^d/(d-1)^{d+1}$. For amenable graphs, it is known that there is a unique Gibbs hard-core distribution for small values of λ (see [70]), while it is conjectured that for large values of λ there is more than one Gibbs distribution.

For Ising models and few other lattice interacting particle systems, a computational phase transition has also been observed. Namely, for some values of the model parameters, the mixing time for the Glauber dynamics on finite graphs is polynomial in the size of the graph, while for other values of the parameter, it is exponential. The location of this phase transition is observed to be at transition between uniqueness and non-uniqueness of the stationary measure in infinite graphs with the same local structure.

We expect that these ideas can be extended to predict a phase transition in our case of the Gibbs hard-core model and process in the continuum. The results of [26] imply that in the infinite Euclidean domain for small values of λ, there is a unique distribution with Papangelou conditional intensity given in eq. 3.2, which is the stationary state of the dynamics of the Continuous Loss Network. While the problem of uniqueness for larger values of λ remains open, it has significant mathematical, physical and computational implications (see [73, 23]).

In [46], more general SBD processes on the Euclidean domain, whose stationary distributions are Gibbs measures were studied, under mild conditions on the interaction potentials. In that work, a semigroup approach is taken to construct the process, and the generator of the process is shown to have a spectral gap, which in particular shows the uniqueness of the stationary measure. The spectral approach for the Gibbs hard-core model was taken in [8].

In the following, we will survey some results mentioned in this section. First, we present and compare two coupling from the past techniques for constructing the stationary regime of the Gibbs hard-core process. The first scheme is similar to the work in Chapter 2, where the decay in density of discrepancies in a simultaneous coupling is used to ensure that the coupling from the past scheme works. The second is a generalization of the framework presented in [34]. Then, we present the results on relaxation times on bounded domains. Specifically, we focus on the work of [8].

3.2 Existence of the Process on $\mathbb{R}^d$

We note that, on bounded domains, the existence of the process with the birth and death rates given in (3.1) is clear, since the total jump rates are finite. In this section, we give a construction of the process on $\mathbb{R}^d$, that will serve as the definition of our process, and will be useful in the construction of the stationary regime.

For the construction of the process on $\mathbb{R}^d$, we require the following alternate viewpoint for the construction of the process on bounded domains. Let $D \subset \mathbb{R}^d$ be a bounded domain. Let $\Phi \in M(D \times \mathbb{R}^+, \mathbb{R}^+)$ be a Poisson point process, with intensity $\lambda\ell \otimes \ell$, and with i.i.d. exponential marks with parameter 1. Let $\eta_0 \in M(D, \times\mathbb{R}^+)$ be an initial condition. We assume that η_0 satisfies the hard-core property that no two points are within a distance 1 from each other.

Any point $x \in \Phi$ is of the form $x = (p_x, b_x, w_x)$, where p_x, b_x and w_x are the position, birth time and the waiting time of the point. Similarly, any point in $x \in \eta_0$ is of the form $x = (p_x, w_x)$. We will treat η_0 as an element of $M(D \times \mathbb{R}^+, \mathbb{R}^+)$, by setting the b_x coordinate to 0 for all $x \in \eta_0$.

On the domain D we may construct the process η_t, by sequentially *processing* the points of Φ according to their birth time. The following algorithm encapsulates this procedure.

- **Data:**
  - a. Φ: A realization of the arrivals.
  - b. η_0: A realization of the initial condition.
  - c. $t \in \mathbb{R}^+$: End time for simulation.

- **Result:** η_t: The final state of the system at time t.

1. Set $t_{old} = 0$.
2. Set $t_{new} = \min(\inf\{b_x : x \in \Phi, b_x > t_{old}\}, \inf\{b_x + w_x : x \in \eta_{t_{old}}\})$. If $t_{new} > t$, quit and return $\eta_{t_{old}}$.
3. If t_{new} is due to arrival of a new particle (first infimum):
   - Let the particle be x.
   - If there is a particle in $\eta_{t_{old}} \cap B_1(p_x)$:
     - Set $t_{old} = t_{new}$. Go to step 2.
   - Else:
     - Set $\eta_{t_{new}} = \eta_{t_{old}} \cup \{x\}$, and $t_{old} = t_{new}$. Go to step 2.
4. If t_{new} is due to the departure of an existing particle (second infimum): Let the particle be x. Set $\eta_{t_{new}} = \eta_{t_{old}} \setminus \{x\}$, and $t_{old} = t_{new}$, and go to step 2.

Using this algorithm, we define a (random) function $h : \Phi \cup \eta_0 \to \{0, 1\}$, that is created as the process is built by the algorithm above, taking the following values. We set $h(x) = 1$ for all $x \in \eta_0$. For $x \in \Phi$, set $h(x) = 1$ if the point x is accepted into the system when it arrives, otherwise, we set $h(x) = 0$. We note that $h(x)$, $x \in \Phi$, satisfies the following recursive property:

$$h(x) = \begin{cases} 1 & \text{if } \{y \in \Phi : b_y < b_x < b_y + w_y, |p_x - p_y| \leq 1, h(y) = 1\} \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{3.3}$$

This recursive property can serve as the definition of the process on $\mathbb{R}^d$ if it can be shown that the recursive property terminates almost surely. From the recursive property, we note that we can compute $h(x)$ if we know all the values of h for the points of $\Phi \cup \eta_0$ in the cylinder $B_1(p_x) \times [0, b_x)$. The following lemma allows us to claim that the recursive definition terminates.

Lemma 3.2.1. *Suppose Φ be a marked Poisson point process as above. Then, there is no infinite sequence $\{x_n\}_{n\in\mathbb{N}} \subset \Phi$ such that $b_{y_i} > b_{y_{i+1}}$ and $|p_{y_i} - p_{y_{i+1}}| \leq 1$ for all $i > 0$.*

Proof. Let $\epsilon > 0$ be fixed. Consider the event $T_\epsilon := \cup_{q\in\mathbb{Q}^+} T_{\epsilon,q}$, where $T_{\epsilon,q}$, $q \in \mathbb{Q}^+$, is the event that the Random Geometric graph (see [58]) obtained by joining any two points in $\{y \in \Phi : q \leq b_y \leq q + \epsilon\}$ whose positions are within a distance 1 from each other does not percolate. From Theorem 3.2 of [58], we know that $\mathrm{P}(T_{\epsilon,q}) = 1$ for every $q \in \mathbb{Q}^+$ if ϵ is small enough. Hence, $\mathrm{P}(T_\epsilon) = 1$ for small enough ϵ. This precludes the presence of an infinite sequence $y_1, y_2 \ldots \in \Phi$ such that $b_{y_i} > b_{y_{i+1}}$ and $|p_{y_i} - p_{y_{i+1}}| \leq 1$, for all $i \in \mathbb{N}$. Indeed, if such a sequence exists then the limit $b = \lim_{i\to\infty} b_{y_i}$ exists and by density of $\mathbb{Q}^+$ in $\mathbb{R}^+$, this event belongs to the set T_ϵ^c. $\square$

This lemma ensures that we can obtain the value of $h(x)$ for any $x \in \Phi$ by recursively applying (3.3). The process can be built using the function h:

$$\eta_t = \{x \in \Phi \cup \eta_0 : b_x \leq t < b_x + w_x, h(x) = 1\}.$$

3.3 Coupling from the Past based on Identity Coupling

In this section we adapt the coupling from the past technique used in Chapter 2 to construct a stationary regime. Consequently, we obtain a lower bound on the critical value, λ_c. In the following sections we present the so-called *Identity coupling* between two processes with different initial conditions. Although, in the identity coupling, two processes never coincide in a finite amount of time almost surely, we can show that they do so locally on any bounded space. We use this result to essentially *patch* the couplings on compact domains, and extend it to the whole space.

3.3.1 Identity Coupling

Consider two different processes, $\{\eta_t^1\}_{t\geq 0}$ and $\{\eta_t^2\}_{t\geq 0}$, starting two different initial conditions, η_0^1 and η_0^2, but driven by the same driving process $\Phi \in M(\mathbb{R}^d \times \mathbb{R}^+, \mathbb{R}^+)$. We call this the identity coupling , since the two processes have the same arrival process.

Assume that the two initial conditions are spatially stationary. As in Section 2.4.2, we consider the following derived processes

$$\begin{aligned} &Z_t := \eta_t^1 \setminus \eta_t^2, \quad A_t := \eta_t^2 \setminus \eta_t^1, \\ &S_t := Z_t \cup A_t, \text{ and } R_t := \eta_t^1 \cap \eta_t^2. \end{aligned}$$

Each of these processes is a spatially stationary process, since they are factors of the process $\Phi_{[0,t]} \cup \eta_0$. The process is S_t will be termed as the process containing *special* points. Let β_{Z_t}, β_{A_t}, β_{S_t} and β_{R_t} denote the densities of the processes $Z_t,\ A_t,\ S_t$ and R_t respectively.

Let $C \subset \mathbb{R}^d$ be compact. The following changes may occur in the values of $\mathrm{E}\, S_t(C)$ in a short interval of time $\delta > 0$.

1. A particle in $C \cap S_t$ may depart from the system. This occurs with probability $\delta S_t(C) + o(\delta)$.

2. A new particle may arrive and get added to the system. This occurs with

probability:

$$\lambda\delta \int_C \mathbb{1}(x \in (Z_t \oplus B_1) \setminus ((R_t \cup A_t) \oplus B_1))$$
$$+ \mathbb{1}(x \in (A_t \oplus B_1) \setminus ((R_t \cup Z_t) \oplus B_1))\mathrm{d}x + o(\delta),$$

where $B_1 = B_1(0)$ is the ball of radius 1 around the point $0 \in \mathbb{R}^d$, and we define the operation $F \oplus G := \{x + y : x \in F, y \in F\}$, for $F, G \subset \mathbb{R}^d$. Thus,

$$\begin{aligned}
\mathrm{E}\, S_{t+\delta}(C) &= \mathrm{E}\, S_t(C) - \delta\, \mathrm{E}\, S_t(C) + \lambda\delta\, \mathrm{E} \int_C \mathbb{1}(x \in (Z_t \oplus B_1) \setminus ((R_t \cup A_t) \oplus B_1)) \\
&\qquad + \mathbb{1}(x \in (A_t \oplus B_1) \setminus ((R_t \cup Z_t) \oplus B_1))\mathrm{d}x + o(\delta) \\
&\leq \mathrm{E}\, S_t(C) - \delta\, \mathrm{E}\, S_t(C) + \lambda\delta\, \mathrm{E} \sum_{x \in S_t \cap C \oplus B_1} \ell(B_1)
\end{aligned} \tag{3.4}$$

Therefore, taking the limit $\delta \to 0$,

$$\ell(C)\frac{\mathrm{d}\beta_{S_t}}{\mathrm{d}t} \leq -\beta_{S_t}\ell(C) + \beta_{S_t}\lambda\ell(C \oplus B_1)\ell(B_1).$$

Since C, is arbitrary, we must have

$$\frac{\mathrm{d}\beta_{S_t}}{\mathrm{d}t} \leq -\beta_{S_t} + \lambda\ell(B_1)\beta_{S_t}.$$

Hence, if $\lambda < \ell(B_1)$, we obtain that the density of the special particles decreases to exponentially quickly to zero.

3.3.2 The Coupling from the Past Construction

In this section, we outline the construction of the stationary regime given that the density of the special points goes exponentially fast to zero.

Let Φ be a doubly infinite Poisson arrival process, i.e., an element of $M(\mathbb{R}^d \times \mathbb{R}, \mathbb{R}^+)$, with intensity $\lambda\ell\times\ell$, and i.i.d. exponentially distributed marks with parameter 1. Let θ_t, $t \in \mathbb{R}$, denote the time-shift operator, such that $\Phi\circ\theta_t(L\times A) = \Phi(L\times(A-t))$, for all $L \subset \mathbb{R}^d$ and $A \subset \mathbb{R}$. The process Φ is stationary and ergodic with respect to $\{\theta\}_{t\in\mathbb{R}}$. Let $\{\eta_t^T\}_{t\geq -T}$, $T \in \mathbb{N}$, be a sequence of processes starting at time $-T$ with empty initial conditions and driven by arrivals from Φ. We claim that $\eta_t^T = \eta_{t+T}^0 \circ \theta_{-T}$.

Let us assume that $\lambda < \ell(B_1)^{-1}$, which is sufficient for exponential decay of The process η_t^1 and η_t^0 are driven by the same process Φ beyond the 0. For any compact $K \subset \mathbb{R}^d$, let

$$\tau^0(K) := \inf\{t > 0 : \eta_s^1|_K = \eta_s^0|_K,\ s \geq t\},$$

and, similarly,

$$\tau^T(K) := \inf\{t > -T : \eta_s^{T+1}|_K = \eta_s^T|_K,\ s \geq t\}.$$

We have the following lemma similar to Lemma 2.4.3, whose proof is skipped.

Lemma 3.3.1. *For all compact $K \subset \mathbb{R}^d$, we have $\mathrm{E}\,\tau^0(K) < \infty$.*

Now, by the definition of the time-shift operator,

$$\tau^T(K) = \tau^0(K) \circ \theta_{-T} - T \implies \tau^T(K) + T = \tau^0(K) \circ \theta_{-T}.$$

Therefore the sequence, $\tau^T(K) + T$ is a stationary and ergodic sequence. By Birkhoff's point-wise ergodic theorem we have, as before

$$\lim_{T\to\infty} \sum_{i=0}^{T} \frac{\tau^i(K) + i}{T} = \mathrm{E}\,\tau^0(K) < \infty, \text{ a.s.}$$

Therefore, can conclude that $\lim_{T\to\infty} \tau_K^T = -\infty$. Thus, for every realization of Φ, any compact set K and $t \in \mathbb{R}$, there exists a $k \in \mathbb{N}$ such that for all $T > k$, $\tau^T(K) < t$. That is, the execution of all processes $\{\eta_s^T\}$, $T > k$, coincides at time t on the compact set K. Then, locally, the following limit is well-defined a.s. on the same probability space:

$$\eta_t := \lim_{T\to\infty} \eta_t^T.$$

As in Section 2.4.2, it can be shown that the process η_t is compatible with the time-shift operator θ_t, and hence is a stationary version of the process.

3.4 Coupling from the Past using Clans of Ancestors

In this section, we outline an application of the Coupling from the past techniques developed in [34] to the Gibbs hard-core process. In this scheme, for the construction of the stationary regime, we start with a marked homogeneous Poisson point process Φ, with intensity $\lambda > 0$ on $\mathbb{R}^d \times \mathbb{R}$. The marks are assumed to be i.i.d. unit exponential random variables. The point and its mark are denoted by $x := (p_x, b_x, w_x) \in \Phi$, where $p_x \in \mathbb{R}^d$, $b_x \in \mathbb{R}$ and $w_x \in \mathbb{R}^+$ are the position, birth time and the waiting time of the particle x, respectively.

We associate the cylinder $B_{1/2}(p_x) \times [b_x, b_x + w_x]$ to the particle x, which represents its influence in space and time. For any $t \in \mathbb{R}$, let ξ_t denote the *free SBD* point process on $\mathbb{R}^d$, obtained by the intersection of $\cup_{x\in\Phi}\{p_x\} \times [b_x, b_x + w_x]$ with $\mathbb{R}^d \times \{t\}$ and ignoring the time coordinate, .i.e.,

$$(\xi_t, t) = (\cup_{x\in\Phi}\{p_x\} \times [b_x, b_x + w_x]) \cap (\mathbb{R}^d \times \{t\}).$$

ξ_t is an SBD process with birth rate $b(x,\gamma) = \lambda$ and death rate $d(x,\gamma) = 1$. We will momentarily give the construction of the Gibbs hard-core process, η_t, by admitting or rejecting particles of ξ_t when they arrive, i.e., ξ_t will be a dominating process of η_t. All process constructed will be factors of the process Φ, and hence will be time-stationary. Thus, this will yield a stationary regime for the Gibbs hard-core process.

For each $x \in \Phi$, we will use the following additional notation

$$R(x) = B_{1/2}(p_x) \times [b_x, b_x + w_x], \text{ Base}(x) = B_{1/2}(p_x), \text{ and Life}(x) = [b_x, b_x + w_x].$$

Denote, for any $x, y \in \Phi$,

$$y \sim x \iff R(x) \cap R(y) \neq \emptyset. \tag{3.5}$$

For an arbitrary $x \in \Phi$, we define the set of first generation ancestors to be

$$A_1^x = \{y \in \Phi : x \sim y \text{ and } b_y < b_x\}. \tag{3.6}$$

These are points that may directly influence whether the particle x is accepted when it arrives. Indeed, if A_1^x is empty, then we are certain that in any simulation that is driven by Φ, the particle x must be accepted into the system. Similarly, we may define the n-th generation ancestors of a point x as

$$A_n^x = \{z \in \Phi : z \in A_1^y \setminus A_{n-1}^x \text{ for some } y \in A_{n-1}^x\}, \quad n \geq 2 \tag{3.7}$$

Also define the *Clan of ancestors* of x to be $A^x = \cup_{n \geq 1} A_1^x$. We say that there is backward oriented percolation, if there exists a particle $x \in \Phi$, with $|A^x| = \infty$.

The existence of a stationary regime is guaranteed if there is no backward oriented percolation almost surely. Indeed, if there is no backward oriented percolation, we can construct η_t by running the forward Gibbs hard-core process on $\bigcup_{x\in\xi_t} A^x$.

We now explore sufficient conditions under which there is no backward oriented percolation. To bound the size of A^x, we also define the parent-child pairs by setting recursively

$$\begin{aligned} P_1^x &= \{x\} \times A_1^x \\ P_n^x &= \{(y,z) : z \in A_1^y, \text{ for some } (w,y) \in P_{n-1}^x\}, n \geq 2. \end{aligned} \tag{3.8}$$

The graph with vertices $\bigcup_n P_n^x$ can be viewed as a branching process, with (y_1, z_1) connected to (y_2, z_2) if an only if $z_1 = y_2$. The size of the process can be stochastically dominated by a Galton-Watson branching process using a standard argument (see [26]), with an off-spring distribution that of $|A_1^x|$.

We now compute the mean of $|A_1^x|$. WLOG, assume that $x = (0, 0, S) \in \Phi$, for some $S \in \mathbb{R}^+$. A particle $y = (p_y, b_y, w_y)$ is in A_1^x if and only if $p_y \in B_1(0)$, $b_y \leq 0$, $b_y + w_y \geq 0$. Thus,

$$\mathrm{E}\,|A_1^x| = \int_{B_1(0)} \int_{-\infty}^{0} \lambda e^t \mathrm{d}t \mathrm{d}x = \lambda \ell(B_1(0)).$$

Therefore, if $\lambda\ell(B_1(0)) < 1$, we obtain that the dominating Galton-Watson branching process does not percolate almost surely, which in-turn implies that there is no backward oriented percolation.

The bound $\lambda < \ell(B_1(0))^{-1}$, is similar to the one obtained in Section 3.3, but a more careful analysis can improve the bound on $|A_n^x|$. Essentially, when using the bound $|A_n^x| \leq |P_n^x|$, we have double counted particles y that are both the level-n and level-$(n-1)$ ancestor of a point x. This can be mitigated by considering 2-levels of ancestors at once, so that if y is a level-1 ancestor, it cannot be a level-2 ancestor. Define

$$\begin{aligned} P_1^x &= \{(0,x,y) : y \in A_1^x\}, \\ P_n^x &= \{(x_1,x_2,x_3) : x_3 \in A_1^{x_2}, x_3 \nsim x_1, \text{ for some } (x_0,x_1,x_2) \in P_{n-1}^x\}. \end{aligned}$$

An element in $(x_1,x_2,x_3) \in P_n^x$ is treated as a child of $(y_1,y_2,y_3) \in P_{n-1}^x$ in the obvious way: if $(x_1,x_2) = (y_2,y_3)$. We still have that

$$|A_n^x| \le |P_n^x| \implies \mathrm{E}\,|A^x| \le \sum_{n\ge 1} \mathrm{E}\,|P_n^x|. \tag{3.9}$$

Let $(0,x,y) \in P_1^x$. Conditioned on the locations y, a point $z \in A_1^y$ is *not* included in P_2^x if and only if $R(x) \cap R(z) \neq \emptyset$, so that $(0,x,z) \in P_1^x$. Given $y \in A_1^x$, the expected number of points that are excluded is equal to:

$$\int_{[B_{1/2}(0)\cap B_{1/2}(p_y)]\oplus B_{1/2}(0)} \int_{-\infty}^{b_y} \lambda e^t \mathrm{d}t\mathrm{d}x = \lambda \ell([B_{1/2}(0)\cap B_{1/2}(p_y)]\oplus B_{1/2}(0)) e^{b_y}.$$

Thus, given P_1^x,

$$\begin{aligned} \mathrm{E}\,|P_2^x| &= [\lambda\ell(B_{1/2}(0)) - \mathrm{E}\,\lambda\ell([B_{1/2}(0)\cap B_{1/2}(p_y)]\oplus B_{1/2}(0))e^{b_y}]|P_1^x| \\ &\le \lambda[\ell(B_1(0)) - \ell(B_{1/2}(0))\,\mathrm{E}\,e^{b_y}]|P_1^x \\ &= \lambda[\ell(B_1(0)) - \frac{1}{2}\ell(B_{1/2}(0))]|P_1^x|, \end{aligned}$$

where we have used that $\ell([B_{1/2}(0)\cap B_{1/2}(p_y)]\oplus B_{1/2}(0)) \ge \ell(B_{1/2}(0))$ and $\mathrm{E}\,e^{b_y} = 1/2$, since $-b_y$ is exponentially distributed with mean one. Similarly, we conclude that $\mathrm{E}\,|P_{n+1}^x| \le \lambda[\ell(B_1(0)) - \frac{1}{2}\ell(B_{1/2}(0))]\,\mathrm{E}\,|P_n^x|$. Hence, there is no backward oriented percolation if $\lambda < [\ell(B_1(0)) - \frac{1}{2}\ell(B_{1/2}(0))]^{-1}$, which improves the bound obtained earlier.

This bound can be improved further by considering more levels of ancestors. In summary, the main idea here is to obtain sufficient conditions to avoid backward oriented percolation, i.e., to allow $|A_1^x| < \infty$ a.s. More careful analysis can yield better bounds for the range of values of λ for which a stationary regime can be constructed by studying the clans of ancestors.

3.5 From Graphs to the Euclidean Domain

Interacting particle systems on graphs form a wide range of probabilistic models studied in statistical physics. In the most general terms, the nearest neighbor spin system on a finite graph $G = (V, E)$ is specified by a finite set S of spin values, a symmetric pair potential $U : S \times S \to \mathbb{R} \cup \{\infty\}$, and a singleton potential $W : S \to \mathbb{R}$. A configuration $\sigma \in S^V$ assigns to each vertex $x \in V$ a spin value $\sigma_x \in S$. The probability of finding the system in a configuration σ is given by the Gibbs distribution

$$\mu(\sigma) \propto \exp\left[-\beta H(\sigma)\right],$$

where β is a model parameter and $H = \sum_{(x,y)\in E} U(\sigma_x, \sigma_y) + \sum_{x\in V} W(\sigma_x)$. The Ising model, for example, is obtained when we have $S = \{+1, -1\}$, $U(\sigma_x, \sigma_y) = \sigma_x\sigma_y$ and $W = 0$.

A Gibbs measure on an infinite graph is defined by taking the limit of finite volume Gibbs measures. A *boundary condition* on a finite domain $\Lambda \subset V$ corresponds to fixing the spins on the complement Λ^c. Given a boundary condition η, we can define the Gibbs measure on Λ as

$$\mu_\Lambda^\eta(\sigma) \propto \exp\left[-\beta\left(\sum_{\substack{(x,y)\in E,\\ \{x,y\}\cap\Lambda\neq\emptyset}} U(\sigma_x,\sigma_y) + \sum_{x\in\Lambda} W(\sigma_x)\right)\right] \mathbb{1}(\sigma_{\Lambda^c} = \eta_{\Lambda^c}).$$

It can be seen that any Gibbs measure on a finite graph satisfies the so-called DLR conditions: For $\Lambda \subset V$, and any $\sigma \subset S^V$,

$$\mu(\cdot|\sigma_{\Lambda^c}) = \mu_\Lambda^\sigma(\cdot).$$

The infinite volume limit Gibbs measure is defined to be a measure where this property holds for every finite subset of vertices and every configuration almost surely. All Gibbs measures can also be obtained by taking the limits of measures of the form μ_Λ^η as Λ approaches V, under different boundary conditions. If the Gibbs measure depends on the boundary conditions considered, i.e., if different boundary conditions yield different Gibbs measures, we say that the system has multiple phases. A phase transition refers to a transition from existence of a unique phase to existence of multiple phases as we change the model parameters. Informally, the existence of multiple Gibbs measures corresponds to the fact that spin configurations on the boundary leaves a non-zero influence at a spin value at a given vertex, as the boundary recedes to infinity.

The Glauber dynamics for an interacting particle system are Markov chains on spin configurations that have the property that their reversible states (i.e., measures with respect to which the dynamics are reversible) coincide with the set of Gibbs states for the model. See [51, 37] for more details.

To illustrate these definitions concretely, let us consider the hard-core model (also called the independent set model) on graphs. The hard-core model is used as a model of lattice gas is statistical physics [37], and in the modeling of communication networks [44]. The hard-core model on a finite graph $G = (V, E)$ is defined as follows. Here, $S = \{0, 1\}$. For a configuration $\sigma \in S^V$, we say that a vertex $x \in V$ is occupied if $\sigma_x = 1$ and unoccupied otherwise. The hard-core model is a probability measure μ on S^V such that

$$\mu(\sigma) \propto \lambda^{\sum_{x\in V} \sigma_x} \mathbb{1}(\sigma \text{ forms an independent set}), \quad \lambda > 0.$$

Thus, $U(1, 1) = \infty$, $U(0, 1) = U(1, 0) = U(0, 0) = 0$ and $W(1) = 1$ and $W(0) = 0$.

We consider the "heat-bath" version of the Glauber dynamics for this model. It is given by a Markov process η_t, $t \geq 0$, where $\eta_t \in S^V$ and η_t is an independent set of V. At every vertex, an independent Poisson clocks clicks at rate λ. At every click of a clock, the spin at the corresponding vertex changes to 1 (occupied) if it does not violate the hard-core condition. Additionally, every vertex flips spin from 1 to 0 at rate 1, independently. We note the similarity in the description of these dynamics to the Gibbs hard-core process considered in this chapter.

The main question this analysis is problem of existence of multiple phases. Dobrushin and Shlosman [22, 20, 21] obtained a widely applicable local criterion which guarantees uniqueness of a Gibbs measure on integer lattices. [74] generalizes these results for other infinite graphs, and moreover, gives coupling based proof for these results. Remarkable connections have also been established between the mixing times of Glauber dynamics on finite graphs and the uniqueness of the Gibbs measure.

First proofs of this property were established using functional analysis techniques, where the spectral gap and log-sobolev constants were used as measures of the rate of mixing in the Markov processes (see [70, 56, 14]). In [74], the authors gave alternate coupling based proofs for this in the attractive case such as the Ising model.

In the following sections, we see that some ideas presented here can be extended to the case of Gibbs hard-core process considered in this chapter. We will characterize the rate at which the Gibbs hard-core process on bounded domains converges to the stationary state in the L^2 sense. We will show that under the spatial mixing condition, the spectral gap of the process is bounded away from 0, uniformly in the volume of the domain and the boundary conditions. This is a summary of results in [8] in the context of the Gibbs hard-core process.

3.5.1 Spectral Gap for the Gibbs Hard-core Process

Let $\Lambda \subset \mathbb{R}^d$ be a bounded set. We consider the Gibbs hard-core process, η_t, on Λ with the boundary condition $\gamma \subset \Lambda^c$. For a given function $f : M(\Lambda) \to \mathbb{R}$, we let

$$D_x^- f(\omega) := f(\omega \setminus \{x\}) - f(\omega),\ D_x^+ f(\omega) := f(\omega \cup \{x\}) - f(\omega),\ \omega \in M(\Lambda), x \in \Lambda.$$

Also, let

$$(D_\Lambda^- f \cdot D_\Lambda^- g)(\omega) := \sum_{x \in \omega} D_x^- f(\omega) D_x^- g(\omega).$$

The generator of the Gibbs hard-core process with boundary conditions γ is given by

$$L_\Lambda^\gamma f(\omega) := \sum_{x \in \omega} D_x^- f(\omega) + \lambda \int_\Lambda \mathbb{1}(\mathrm{dist}(x, \omega \cup \gamma) > 1) D_x^+ f(\omega) \mathrm{d}x, \qquad \omega \in M(\Lambda), \tag{3.10}$$

where $\operatorname{dist}(x,B) := \inf\{|x-y| : y \in B\}$, $B \subseteq \Lambda^c$. In the case of empty boundary conditions, we denote $L_\Lambda^\emptyset = L_\Lambda$.

This process is reversible and the stationary measure is given by the following density with respect to the unit intensity Poisson point process on Λ.

$$\mu_\Lambda^\gamma(\mathrm{d}\omega) = (Z_\Lambda^\gamma)^{-1} e^{\ell(\Lambda)} \lambda^{|\omega|} \mathbb{1}(\operatorname{dist}(x, \omega \cup \gamma) > 1) P_\Lambda(\mathrm{d}\omega), \tag{3.11}$$

where $P_\Lambda(\mathrm{d}x_1, \ldots, \mathrm{d}x_k) = \frac{e^{-\ell(\Lambda)}}{k!} \mathrm{d}x_1 \cdots \mathrm{d}x_k$ is the density of the Poisson point process, $|\omega|$ denotes the number of points in ω, and Z_Λ^γ is a normalizing constant. For the domain of the generator we set

$$\mathcal{D}_0(L_\Lambda^\gamma) = \{f \in L^2(\mu_\Lambda^\gamma) : \exists M \in \mathbb{N}, |f| \leq M \text{ and } f(\omega) = 0 \text{ when } |\omega| > M\}.$$

The Dirichlet form associated with L_Λ^γ is given by the two-form

$$\mathcal{E}_\Lambda^\gamma(f,g) := \langle (-L_\Lambda^\gamma) f, g \rangle_{L^2(\mu_\Lambda^\gamma)}, \qquad f, g \in \mathcal{D}_0(L_\Lambda^\gamma). \tag{3.12}$$

and we let $\mathcal{E}_\Lambda^\gamma(f) := \mathcal{E}_\Lambda^\gamma(f,f)$. We also define the covariance two-form as

$$\mu_\Lambda^\gamma(f,g) := \mu_\Lambda^\gamma(fg) - \mu_\Lambda^\gamma(f)\mu_\Lambda^\gamma(g). \tag{3.13}$$

Let $P_t^{\Lambda,\gamma}$ denote the probability semigroup generated by the dynamics described above.

Our main result in this section, is that under a spatial mixing condition, the spectral gap of the operator L_Λ^γ is bounded away from zero uniformly in the volume of Λ and the boundary condition γ. We will develop the terminology for this result and the result itself in several steps. First, we need the following lemma.

Lemma 3.5.1. *For any $f, g \in \mathcal{D}_0(L_\Lambda^\gamma)$, we have $\mathcal{E}_\Lambda^\gamma(f,g) = \mu_\Lambda^\gamma(D_\Lambda^- f \cdot D_\Lambda^- g)$. Thus, L_Λ^γ is symmetric on $\mathcal{D}_0(L_\Lambda^\gamma)$.*

Proof. We have

$$\begin{aligned}
&\mathcal{E}_\Lambda^\gamma(f,g) \\
&= -\mu_\Lambda^\gamma(D_\Lambda^- f \cdot D_\Lambda^- g) \\
&= \mu_\Lambda^\gamma\left[-\sum_{x\in\omega} D_x^- f(\omega) g(\omega\setminus\{x\}) + \lambda\int_\Lambda \mathbb{1}(\mathrm{dist}(x,\omega)>1) D_x^+ f(\omega) g(\omega)\mathrm{d}x\right] \\
&= 0,
\end{aligned}$$

where in the last step we have used the Georgii-Nguyen-Zessin formula (see [36]). $\square$

The spectral gap of the operator L_Λ^γ is defined as

$$\mathrm{gap}(L_\Lambda^\gamma) := \inf_{f\in\mathcal{D}_0(L_\Lambda^\gamma)} \frac{\mathcal{E}_\Lambda^\gamma(f,f)}{\mu_\Lambda^\gamma(f,f)},$$

where $\mathcal{E}_\Lambda^\gamma$ and μ_Λ^γ are defined in (3.12) and (3.13). Further, we say that a Poincaré type inequality holds if there exists G such that

$$\mu_\Lambda^\gamma(f,f) \le G\mathcal{E}_\Lambda^\gamma(f,f), \quad f\in\mathcal{D}_0.$$

The Poincaré inequality is equivalent to the following two statements.

1. $\mathrm{gap}(L_\Lambda^\gamma)^{-1} \le G$.
2. $\|P_t^{\Lambda,\gamma} f - \mu_\Lambda^\gamma f\|_{L^2(\mu_\Lambda^\gamma)} \le (\mu_\Lambda^\gamma(f,f))^{1/2} e^{-t/G}$, for all $f\in L^2(\mu_\Lambda^\gamma)$.

Thus, the probability semigroup converges to the stationary measure in the L^2 sense if the Poincaré inequality holds. The *relaxation time* of the Markov process (see [50]) is less that G.

For a subset $\Lambda \subset \mathbb{R}^d$, let $\mathcal{M}_\Lambda$ denote the σ-algebra on $M(\mathbb{R}^d)$ generated by the maps $\mu \mapsto \mu(A)$, $A \subset \Lambda$. A function f is measurable with respect to $\mathcal{M}_\Lambda$ (denoted $f \in \mathcal{M}_\Lambda$) only if $f(\omega) = f(\omega \cap \Lambda)$. The spatial mixing property we require for Gibbs measures is the following.

Definition 3.5.1 (Spatial mixing property)**.** We say that a collection of Gibbs measures is spatially mixing if there exists constants α and m such that, for $\Lambda_f \subset \Lambda$,

$$|\mu_\Lambda^\gamma(f) - \mu_\Lambda^\omega(f)| \leq \alpha \mu_\Lambda^\gamma(|f|) e^{-m \operatorname{dist}(\Lambda_f, \gamma \triangle \omega)},$$

for all $\gamma, \omega \subset \Lambda^c$, for all $f \in \mathcal{M}_{\Lambda_f}$.

Under the above spatial mixing assumption, we can prove the following theorem

Theorem 3.5.2. *If the spatial mixing property above holds, then there exists a constant G, such that for all $\Lambda \subset \mathbb{R}^d$ and $\gamma \subset \Lambda^c$,*

$$\mu_\Lambda^\gamma(f, f) \leq G \mathcal{E}_\Lambda^\gamma(f, f), \qquad \textit{for all } f \in \mathcal{D}_0(L_\Lambda^\gamma). \tag{3.14}$$

There are two steps in the proof of the above theorem. First, we prove a trivial bound on the spectral gap, that depends on the volume of the domain Λ. Next, using the spectral mixing property, we show that we can join two domains of equal size without increasing the spectral gap of the generator. This part of the argument is the standard in the theory of lattice spin systems (see Theorem 4.5 of [56]), and we adapt the same strategy to our system.

We first give the following trivial bound on the spectral gap.

Proposition 3.5.3. *For all $\lambda > 0$, $\Lambda \subset \mathbb{R}^d$ bounded, and $\gamma \subset \Lambda^c$, we have*

$$\mu_\Lambda^\gamma(f, f) \leq 2e^{\lambda\ell(\Lambda)}\mathcal{E}_\Lambda^\gamma(f, f), \qquad \forall f \in \mathcal{D}_0(L_\Lambda^\gamma).$$

Hence, gap $\geq (2e^{\lambda\ell(\Lambda)})^{-1}$.

Proof. Using the product coupling, the covariance may be written as

$$\mu_\Lambda^\gamma(f, f) = \frac{1}{2}(Z_\Lambda^\gamma)^{-2} \sum_{n,m=0} \frac{\lambda^{m+n}}{n!m!} \int_{\Lambda^n} \int_{\Lambda^m} \mathbb{1}(|x_i - x_j| > 1)\mathbb{1}(|y_i - y_j| > 1) \\ \times [f(x) - f(y)]^2 \mathrm{d}x\mathrm{d}y.$$

$f(x) - f(y)$ can be written as a telescopic sum:

$$f(x_1, \ldots, x_n) - f(y_1, \ldots, y_m) = -\sum_{k=1}^{n} D_{x_k}^- f(x_1, \ldots, x_k) + \sum_{k=1}^{m} D_{y_k}^- f(y_1, \ldots, y_k),$$

and by Cauchy-Schwartz inequality, we have:

$$\frac{1}{2}[f(x) - f(y)]^2 \leq n \sum_{k=1}^{n} [D_{x_k}^- f(x_1, \ldots, x_k)]^2 + m \sum_{k=1}^{m} [D_{y_k}^- f(y_1, \ldots, y_k)]^2.$$

Therefore,

$$
\begin{aligned}
\mu_\Lambda^\gamma(f,f) &\le 2(Z_\Lambda^\gamma)^{-1}\sum_{n=1}^\infty \int_{\Lambda^n} \frac{\lambda^n}{(n-1)!}\mathbb{1}(\delta(x_i,x_j)>1)\sum_{k=1}^n [D_{x_k}^- f(x_1,\dots,x_k)]^2 \mathrm{d}x \\
&= 2(Z_\Lambda^\gamma)^{-1}\sum_{k=1}^\infty \sum_{n=k}^\infty \int_{\Lambda^n} [D_\Lambda^- f\cdot D_\Lambda^- f(x_1,\dots,x_k)]\frac{\lambda^n}{(n-1)!k}\mathbb{1}(|x_i-x_j|>1)\mathrm{d}x \\
&\le 2(Z_\Lambda^\gamma)^{-1}\sum_{k=1}^\infty \int_{\Lambda^k} [D_\Lambda^- \cdot D_\Lambda^- f(x_1,\dots,x_k)]\frac{\lambda^k}{k!}\mathbb{1}(|x_i-x_j|>1)\mathrm{d}x \\
&\qquad\qquad \times \sum_{n=k}^\infty \int_{\Lambda^{n-k}} \frac{\lambda^{n-k}(k-1)!}{(n-1)!}\mathrm{d}y \\
&\le 2\mathcal{E}(f,f)\sum_{n=0}^\infty \frac{\lambda^n|\Lambda|^n}{n!} \\
&= 2e^{\lambda|\Lambda|}\mathcal{E}_\Lambda^\gamma(f,f).
\end{aligned}
$$

□

3.5.2 Proof of Theorem 3.5.2

The main idea of the proof is to show that the spectral gap stays almost the same when we double the volume. Let $\Lambda \subset \mathbb{R}^d$ be a rectangular box with the longest side of length L in the direction e_1. Write $\Lambda = A \cup B$, where A and B are two rectangles of roughly the same size with a small overlap of order $\sqrt{L}$ in the direction e_1.

Using the spatial mixing bound we obtain the following inequality. For any $f \in \mathcal{M}_{A^c}$ bounded, and any $\gamma \in \Lambda^c$,

$$
\begin{aligned}
\|\mu_B^{(\cdot)} f - \mu_\Lambda^\gamma f\|_\infty = \|\mu_B^{(\cdot)} f - \mu_\Lambda^\gamma \mu_B^{(\cdot)} f\|_\infty &\le \sup_{\omega,\tau\in M(\Lambda)} |\mu_B^\omega f - \mu_B^\tau f| \\
&\le \alpha\|f\|_1 e^{-m\,\mathrm{dist}(\Lambda\setminus A,\Lambda\setminus B)} \le \alpha\|f\|_\infty e^{-m\sqrt{L}}.
\end{aligned}
$$

Thus, the operator T_B that maps $f \in \mathcal{M}_{A^c}$ to $\mu_B^{(\cdot)}(f) - \mu_\Lambda^\gamma(f) \in \mathcal{M}_{B^c}$ is a bounded operator from $L^\infty(M(\Lambda), \mathcal{M}_{A^c}, \mu_\Lambda^\gamma)$ to $L^\infty(M(\Lambda), \mathcal{M}_{B^c}, \mu_\Lambda^\gamma)$, with

$$\|T_B\|_{L^1\infty \to L^\infty} \le \alpha e^{-m\sqrt{L}}.$$

Similarly, the operator T_A that maps $g \in \mathcal{M}_{B^c}$ to $\mu_A^{(\cdot)}(g) - \mu_\Lambda^\gamma(g) \in \mathcal{M}_{A^c}$ is a bounded operator from $L^\infty(M(\Lambda), \mathcal{M}_{B^c}, \mu_\Lambda^\gamma)$ to $L^\infty(M(\Lambda), \mathcal{M}_{A^c}, \mu_\Lambda^\gamma)$, with

$$\|T_A\|_{L^1\infty \to L^\infty} \le \alpha e^{-m\sqrt{L}}.$$

Now, consider $f \in L^\infty(M(\Lambda), \mathcal{M}_{A^c}, \mu_\Lambda^\gamma)$ and $g \in L^1(M(\Lambda), \mathcal{M}_{B^c}, \mu_\Lambda^\gamma)$. We have

$$\begin{aligned}
\mu_\Lambda^\gamma(T_B f)g &= \mu_\Lambda^\gamma[(\mu_B^{(\cdot)} f - \mu_\Lambda^\gamma f)g] = \mu_\Lambda^\gamma \mu_B^{(\cdot)}(fg) - \mu_\Lambda^\gamma f \mu_\Lambda^\gamma g \\
&= \mu_\Lambda^\gamma(fg) - \mu_\Lambda^\gamma f \mu_\Lambda^\gamma g = \mu_\Lambda^\gamma \mu_A^{(\cdot)}(fg) - \mu_\Lambda^\gamma f \mu_\Lambda^\gamma g \\
&= \mu_\Lambda^\gamma[f(\mu_A^{(\cdot)} g - \mu_\Lambda^\gamma g)] = \mu_\Lambda^\gamma[f T_A g].
\end{aligned}$$

Thus, T_A as a map from $L^1 \to L^1$ is also bounded with $\|T_A\|_{L^1 \to L^1} \le \alpha e^{-m\sqrt{L}}$. Similarly, $\|T_B\|_{L^1 \to L^1} \le \alpha e^{-m\sqrt{L}}$. By the Riesz-Thorin interpolation theorem

$$\|T_A\|_{L^2 \to L^2} \le \alpha e^{-m\sqrt{L}} \quad \text{and} \quad \|T_B\|_{L^2 \to L^2} \le \alpha e^{-m\sqrt{L}}.$$

Let $\varepsilon = \alpha e^{-m\sqrt{L}}$. Now, for $f \in L^2(\mu_\Lambda^\gamma)$ with $\mu_\Lambda^\gamma f = 0$, we have

$$\begin{aligned}
\mu_\Lambda^\gamma(f, f) &= \mu_\Lambda^\gamma(f^2) - \mu_\Lambda^\gamma(f \mu_A^{(\cdot)} f) + \mu_\Lambda^\gamma(f \mu_A^{(\cdot)} f) \\
&= \mu_\Lambda^\gamma(\mu_A^{(\cdot)}(f, f)) + \mu_\Lambda^\gamma(f \mu_A^{(\cdot)} f).
\end{aligned} \tag{3.15}$$

The second term can be written as

$$\begin{aligned}
\mu_\Lambda^\gamma(f \mu_A^{(\cdot)} f) &= \mu_\Lambda^\gamma[(f - \mu_B^{(\cdot)} f)\mu_A^{(\cdot)} f] + \mu_\Lambda^\gamma(\mu_B^{(\cdot)} f \mu_A^{(\cdot)} f) \\
&\le [\|f - \mu_B^{(\cdot)} f\|_2 + \varepsilon \|f\|_2] \|\mu_A^{(\cdot)} f\|_2 \\
&= [\|f - \mu_B^{(\cdot)} f\|_2 + \varepsilon \|f\|_2] \mu_\Lambda^\gamma(f \mu_A^{(\cdot)} f)^{1/2}.
\end{aligned} \tag{3.16}$$

(3.16) implies that

$$\mu_\Lambda^\gamma(f\mu_A^{(\cdot)}f) \le \|f-\mu_B^{(\cdot)}f\|_2^2 + 2\varepsilon\|f\|_2\|f-\mu_B^{(\cdot)}f\|_2 + \varepsilon^2\|f\|_2^2.$$

Since $\|f-\mu_B^{(\cdot)}f\|_2^2 = \mu_\Lambda^\gamma(\mu_B^{(\cdot)}(f,f)) \le \|f\|_2 = \mu_\Lambda^\gamma(f,f)$, we have

$$\mu_\Lambda^\gamma(f\mu_A^{(\cdot)}f) \le \mu_\Lambda^\gamma(\mu_B^{(\cdot)}(f,f)) + [2\varepsilon+\varepsilon^2]\mu_\Lambda^\gamma(f,f). \tag{3.17}$$

If L is large enough so that $1-2\varepsilon-\varepsilon^2>0$, from (3.15) and (3.17) we obtain

$$\mu_\Lambda^\gamma(f,f) \le (1-2\varepsilon-\varepsilon^2)^{-1}\mu_\Lambda^\gamma[\mu_A^{(\cdot)}(f,f)+\mu_B^{(\cdot)}(f,f)].$$

Thus, there exists a constant $c_1>0$ such that, for large enough L,

$$\mu_\Lambda^\gamma(f,f) \le (1+c_1e^{-m\sqrt{L}})\mu_\Lambda^\gamma[\mu_A^{(\cdot)}(f,f)+\mu_B^{(\cdot)}(f,f)]. \tag{3.18}$$

We can now upper bound $\mu_\Lambda^\gamma[\mu_A^{(\cdot)}(f,f)+\mu_B^{(\cdot)}(f,f)]$ in terms of the spectral gaps on A and B. Let, for $V\subset\mathbb{R}^d$,

$$G_V := \sup_{\omega\subset V^c} \operatorname{gap}(L_V^\omega)^{-1}.$$

We have

$$\begin{aligned}\mu_\Lambda^\gamma[\mu_A^{(\cdot)}(f,f)+\mu_B^{(\cdot)}(f,f)] &\le (G_A\vee G_B)\mu_\Lambda^\gamma[\mu_A^{(\cdot)}(|D_A^-(f)|^2)+\mu_B^{(\cdot)}(|D_B^-(f)|^2)]\\ &= (G_A\vee G_B)\mu_\Lambda^\gamma[|D_A^-(f)|^2+|D_B^-(f)|^2]\\ &= (G_A\vee G_B)[\mathcal{E}_\Lambda^\gamma(f,f)+\mu_\Lambda^\gamma|D_{A\cap B}^-(f)|^2]\end{aligned} \tag{3.19}$$

From (3.18) and (3.19), we obtain

$$\mu_\Lambda^\gamma(f,f) \le (1+c_1e^{-m\sqrt{L}})(G_A\vee G_B)[\mathcal{E}_\Lambda^\gamma(f,f)+\mu_\Lambda^\gamma|D_{A\cap B}^-(f)|^2].$$

The term $\mu_\Lambda^\gamma|D^-_{A\cap B}(f)|^2$ in the RHS can be bounded by, but that yields gives an upper bound of $2(1+c_1e^{-m\sqrt{L}})(G_A \vee G_B)$ on G_Λ. Instead, we average over $\lfloor L^{1/3}\rfloor/4$ number of partitions $\{A_i, B_i\}_{i=1}^{\lfloor L^{1/3}\rfloor/4}$, such that $A_i \cup B_i \cup A_j \cup B_j = \emptyset$. Then, we have

$$\begin{aligned}\mu_\Lambda^\gamma(f,f) &\le (1+c_1e^{-m\sqrt{L}})(1+\lfloor 1/L^{1/3}\rfloor)\sup_i(G_{A_i}\vee G_{B_i})[\mathcal{E}_\Lambda^\gamma(f,f)]\\ &\le \left(1+\frac{8}{L^{1/3}}\right)\sup_i(G_{A_i}\vee G_{B_i})\mathcal{E}_\Lambda^\gamma(f,f),\end{aligned}$$

for all $L > L_1$, for some L_1 large enough. Thus, $G_\Lambda \le \left(1+\frac{8}{L^{1/3}}\right)\sup_i(G_A \vee G_B)$.

Now, if Λ is a rectangle, with the largest side much greater that L_1, successively partition Λ loosely into smaller rectangles by adding a factor of $\left(1+\frac{8}{L^{1/3}}\right)$ to it each time. After d such steps, it is easy to see that we have reduced the largest side by at least a factor of $3/4$. Thus, after a finite number of steps, we will obtain all rectangles in the partition with sides less that $2L_1$. So, from the above bounds, and Proposition 3.5.3,

$$G_\Lambda \le G_{2L_1}\prod_{i=1}^{\infty}\left(1+\frac{8}{(4/3)^{i/3d}L_1^{1/3}}\right) < \infty,$$

where $G_{2L_1} = 2e^{\lambda(2L_1)^d}$. Thus, the spectral gap is bounded away from zero uniformly in the size of the rectangle. This completes the proof of Theorem 3.5.2.

In conclusion, we note a sufficient condition for spatial mixing property in Definition 3.5.1 to hold. Using the so called Cluster expansion technique (see Lemma 4 of [69]), it can be shown that if $\lambda < 1/(3e\ell(B_1(0)))$, the spatial mixing condition holds.

3.6 Concluding Remarks and Future Work

In this chapter we surveyed a few techniques to study the infinite domain Gibbs hard-core process. We noted two coupling based techniques to show the existence and convergence to a stationary regime, when the arrival rate is small enough. We also noted precise sufficient conditions under for making each of those techniques work. We claim that a more careful analysis of these sufficient conditions can yield better range of values for λ.

In the third approach, we surveyed a function-analytic technique to obtain existence and convergence to stationarity. Here, we noted that the spatial mixing condition for the Gibbs measures on bounded domains is a sufficient condition for existence of a unique Gibbs measure on the infinite domain. We then prove that under the spatial mixing condition, the spectral gap on the dynamics on the bounded domains is bounded away from zero uniformly in the volume of the domain. This also implies that the spectral gap of the infinite domain dynamics is also positive. This yields exponential convergence to stationarity in the L^2 sense.

Function analytic tools have been utilized to obtain stronger results in discrete setting, in particular for the Glauber dynamics of the independent set model on graphs. It was shown that, under the spatial mixing condition, the log-sobolev constant of the generator is finite that scales as a polynomial in the number of vertices in the domain (see [70, 56, 14]). This implies that the mixing time of the dynamics is at most a polynomial in the number of the vertices in the domain. We conjecture that the mixing time of the Gibbs hard-core dynamics is at most polynomial in the volume of the domain. One approach to the proving this conjecture would be to discretize the domain Λ into a suitably fine lattice, and approximate the dynamics with a birth-death process on this graph. As we improve the approximation by taking a smaller grid size, the number of lattice points we need to consider increases. We expect that the effect of increasing the lattice points on the mixing time is mitigated by the smaller arrival rate at each lattice point and the larger interaction radius of points when we perform such a scaling. We leave this analysis for future work.

Chapter 4

The Spatial Matching Process - the First-in-First-Match Case[1]

4.1 Introduction

Let D be a metric space, with complete metric d, and let λ be a Radon measure defined over it. Suppose for now that D is compact, so that $\lambda(D) < \infty$. In this chapter, we study the time-evolution of a continuous time stochastic Markov jump process $\{\eta_t\}_{t\geq 0}$ whose state is defined by an ordered *configuration* of two types of points in D. The two types are assigned colors *red* and *blue*, and referred in short by the letters R and B respectively. A configuration here refers to a locally finite collection of points. When D is compact, a configuration consists of finite number of points.

The process evolves over the space of ordered configurations on $D \times \{\mathtt{R}, \mathtt{B}\}$ as follows. New particles of each type arrive according to an independent Poisson point process on $D \times \mathbb{R}^+$ with intensity $\lambda \times \ell$, where ℓ is the Lebesgue measure on $\mathbb{R}^+$. Suppose, for instance, that a red particle arrives at time $t > 0$, at location $x \in D$. Then we look for the first blue particle in the ordered sequence η_{t-} whose distance to location x is less than 1. If there is such a particle, $p \in \eta_{t-}$, the new state

[1] The results and analysis presented in this chapter were originally published by the author in [55].

η_t is obtained by removing the particle p from η_{t-}, while keeping the order of the remaining elements fixed. If there is no such particle, then the new state is obtained by adding a particle, with location x and mark $\mathtt{R}$, to η_{t-}. In this case, the order within the elements of η_{t-} is preserved and the new particle is placed at the end of the sequence η_{t-}. The arrival of a blue particle is handled similarly. Additionally, any particle in the configuration is removed at a constant rate $\mu > 0$, while preserving the order of the remaining particles. Note that if η_0 is the empty configuration, then the ordering of particles in η_t is simply the order in which those particles have arrived in the system. We will call this the First-in-first-match (FIFM) spatial matching process.

Figure 4.1 gives an illustration for the above dynamics.

4.1.1 Motivation and Previous Work

The motivation for studying this problem comes from modern shared-economy markets, where individuals engage in monetized exchange of goods that are privately owned in a via peer-to-peer marketplace. Examples of such marketplaces include ride-sharing networks, such as Uber or Lyft, and renewable energy networks with distributed generation of power. Here, consumers and producers can be viewed as individuals distributed in an abstract space, who engage in a transaction with others in close proximity. The abstract space could model factors such as location, product preferences, price and willingness to pay, etc. In the example of ride-sharing network, the position of an individual would correspond to its physical location, while in a renewable energy network, the position could model a combination of physical

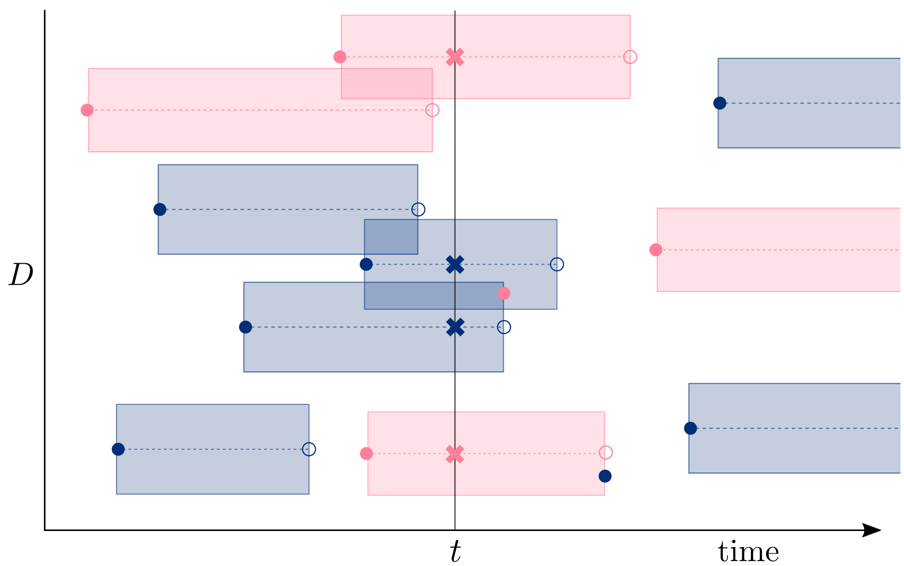


Figure 4.1: An illustration of the FIFM spatial matching process. The vertical dimension represents the set D. The rectangles represent the lifetimes of particles in the system – so, the vertical dimension of the rectangle represents the spatial range of interaction of a particle, a solid disk to the left of a rectangle represent its arrival, and a hollow circle at the right represents its departure. The set of particles present in η_t are marked by crosses; these are those particles whose rectangles intersect the "vertical line" at time t.

location and price. In this study, our goal is to comment on the spatial distribution of individuals in the long-run, under a well-defined matching scheme such as the one described in the introduction.

The underlying dynamics in our model can be viewed from a queuing theoretic viewpoint. Most queuing theoretic models study systems where there is an inherent asymmetry between customers and servers. Customers are usually transient agents that arrive with some load, and depart on being processed. Servers meanwhile are present during the whole life-time of the study of the stochastic process, and serve the customers according to a given policy. In the literature, there are only a few examples of queuing systems where customers and servers are treated as symmetric agents that *serve* each other. The double-ended queuing model discussed in [43] studies a model for a taxi-stop where taxis and customers arrive independently according to two Poisson arrivals. If a taxi (or a customer) arrives at the taxi-stop and finds a waiting customer (taxi) waiting, then it matches instantly, using say a first-come-first-serve (FCFS) policy, and both agents depart. Otherwise, the taxi (customer) waits until it is matched with a customer (taxi) that arrives later.

The FCFS bipartite matching model that was introduced in [13], and later studied in some generality in [1], is another such model. In this model, the customers and servers belong to finite sets of types, C and S respectively, which determine whom they can be matched to. The compatibility of matches between the various types of customers and servers is expressed in terms of a bipartite graph $G = (C, S, \mathcal{E})$, where $\mathcal{E} \subset C \times S$. The process, $\{\eta_t\}_{t\in\mathbb{N}}$, is the ordered list of unmatched customers and servers arriving before time $t \in \mathbb{N}$. At each time $t \in \mathbb{N}^+$, one customer, $c_t \in C$, and one server, $s_t \in S$, arrive to the system. Here, $\{c_t\}_{t\in\mathbb{N}^+}$ and $\{s_t\}_{t\in\mathbb{N}^+}$ are independent sequences of i.i.d. random elements of C and S, with distributions α and β, respectively. η_t is obtained from η_{t-1}, c_t and s_t by matching c_t and s_t from elements in η_t, if possible, using the FCFS policy, and removing the matched pairs. This model is called the *FCFS bipartite matching model*. In this series of works ([13, 2, 1]), the authors derive a product form distribution for the steady state under the so-called *complete resource pooling* condition:

$$\alpha(A) < \sum_{(x,y)\in A\times S\subseteq E} \beta(y),\ \forall A \subsetneq C,$$

or equivalently,

$$\beta(A) < \sum_{(x,y)\in C\times A\subseteq E} \alpha(x),\ \forall A \subsetneq S.$$

The authors also provide expressions for performance measures in the steady state, such as the matching rates between certain type of pairs, and waiting times of agents. These expressions are computationally hard to evaluate, owing to the hardness in computing the normalizing constant in the product form distribution.

We now briefly discuss variants of the FCFS bipartite matching model. Bušić et.al ([12]) generalize the bipartite matching model by dropping independence of arriving types and considering other matching policies. Büke and Chen [11] study a model where the matching policy is probabilistic. In their model, when a customer (server) arrives in a system, it looks at the possible matches and independently of everything else, selects one using a probability distribution. There is also a positive probability of not finding any suitable server (customer), in which case it starts to wait for a compatible server (customer). They also consider models where the users are impatient and may depart if they are not matched by a certain time. An exact analysis of these models becomes quite intractable and in [11] the authors study the fluid and diffusive scaling approximations of these systems.

The model we consider in this paper is essentially a continuous time and continuous space version of the model studied in [1], with the added feature that particles may also depart on their own due to a loss of patience. This spatial matching model is related to the FCFS bipartite matching model in the following sense. Just like the classes of customers and servers in that model, in our model, we still have two classes, the red and blue particles; each particle has a location in D which is akin to the types within a class; and two particles are supposed to be compatible in the sense of the FCFS bipartite matching model, if they are within a distance one from each other.

In this chapter, in the case when D is compact, we derive a product form characterization of the steady state distribution of the process we consider. The analysis needed to obtain this product form distribution is an extension of the analysis in [1] to the continuum. We guess the reversed process and the steady state, and then check the local balance conditions to get the product form distribution in Theorem 4.2.2.

The two particle Widom-Rowlinson (WR) model is related to the distribution of the unordered configuration, $\tilde{\eta}$, in the steady state of our model. The one and the two particle WR models were defined in [75] as a mathematical model for the study of liquid-vapor phase transition in physical systems. The two particle WR model is a point process that consists of two types of particles, $\mathtt{R}$ and $\mathtt{B}$, as in the steady state of our process. This model on a compact domain can be described as the union of two Poisson point processes with intensities $\lambda_{\mathtt{R}}$ and $\lambda_{\mathtt{B}}$, conditioned on the event that there are no two particles of opposite types within a distance one from each other. On the infinite Euclidean domain, $\mathbb{R}^d$, it is defined as a Markov random field (see [72]), with Papangelou conditional intensity $\varphi((x, \mathtt{R}), \eta) = \lambda_{\mathtt{R}} \mathbb{1}(d(x, \eta^{\mathtt{B}}) > 1)$ and

$\varphi((x,\mathtt{B}),\eta) = \lambda_{\mathtt{B}}\mathbb{1}(d(x,\eta^{\mathtt{R}}) > 1)$, where $\eta^{\mathtt{R}}$ and $\eta^{\mathtt{B}}$ are the collection of red and blue points of η. The single particle WR model is obtained by marginalizing over one of the particles in the two particle WR model. The two particle WR model is interesting as it is the first continuum Markov random process where a phase transition has been rigorously established. It was basically established that, in the phase diagram, along the line $\lambda_{\mathtt{R}} = \lambda_{\mathtt{B}} = \lambda$, symmetry is broken when λ is large enough. This was first shown in [66] using an adaptation of Peierl's argument. [15, 39] independently give modern self-contained proofs of this phenomenon using percolation based arguments. The key ingredient in the proof in [15] is the observation that when we disregard the types of the particles, the resulting model, called the Gray WR model, is the continuum version of the random cluster model. The Gray model, in particular, satisfies the FKG property (with the usual lattice structure) and the corresponding positive association inequalities, that are crucial in showing existence of a percolation thresholds, and consequently a phase transition in this model.

Contrary to the two particle WR model, we believe that in our model, the unordered collection of the points in the steady state is not a Markov points process. The definition of a Markov point process (see Chapter 2 of [72]) requires, first, that there exists a symmetric reflexive relation $\sim$ over the domain, and second, that the Papangelou conditional intensity at a point x depend on the configuration η only through the points in η that are related to the x by $\sim$. In our case, we are unable to show the existence of such a symmetric relation. However, we are able to show that the Papangelou conditional intensity at a point depends on the clusters of overlapping unit balls that intersect with the unit ball around the point.

In spite of this limitation, we show that the point process satisfies an FKG lattice property, with a specific lattice structure, similar to one satisfied by the two particle WR model (see Section 2 of [15]). The lattice structure as follows: we say $\eta > \eta'$ if and only if $\eta^{\mathtt{R}} \supset \eta'^{\mathtt{R}}$ and $\eta^{\mathtt{B}} \subset \eta'^{\mathtt{B}}$. The resulting positive association inequality can be used in conjunction with the results in [9], to prove that the points of the same type are weakly-super Poissonian. This is interesting since any exact analysis of the clustering of the steady state from its product form distribution is prohibitively hard as there is no closed form expression for its normalizing constant. In fact, in discrete systems, it is a $\sharp P$-complete problem to compute the normalizing constant ([2]).

We also consider the same matching dynamics in the infinite Euclidean domain $\mathbb{R}^d$. In this regime, using coupling from the past based arguments, we give a formal definition and a construction of the process, and show that there exists a stationary regime for this process. The existence of the stationary regime is obtained using certain coupling from the past idea is similar to the one used in Chapter 3.

In the following sections, we will discuss the notation required to formally define our model. Other notation will be required as we go along – see Appendix A.7 for a table of notation. We begin Section 4.2 with the formal definition of the model in a compact domain. In Section 4.2.1, we give a coupling based argument that the steady state exists and is unique, and in Section 4.2.2, we present the product form distribution for this steady state. Then, in Section 4.2.3, we introduce and prove the FKG lattice property satisfied by the unordered version product form distribution. We then proceed to study the model in the infinite Euclidean domain, in Section 4.3.

In Section 4.3.1, we give a construction and in Section 4.3.3, we give the construction of the stationary regime.

4.1.2 Notation

Let S be any metric space, endowed with a Radon measure, $\lambda_S(\cdot)$. From the Section 1.1.1, we use $M(S)$ to denote the space of simple counting measures on S, and $\mathcal{M}(S)$ to denote the corresponding σ-algebra.

Every particle in our model also carries information about its patience. To encode this, we need the notion of a marked counting measure. A marked simple counting measure on S, with marks in a space K, an l.c.s.h space, is denoted by $M(S, K)$.

We will also require the definition of the space of locally-finite totally-ordered collection of points in the space S, $O(S)$. For any $\xi \in O(S)$, the order within the elements of ξ will be denoted by $<_\xi$. The order will be used to indicate the priority of the particles when matching with other particles. So, if the state of the system is $\xi \in O(S)$ and an incoming point x is compatible with both $y_1 <_\xi y_2$, then it prefers y_1 over y_2. $O(S)$ has a natural projection onto $M(S)$, obtained by dropping the order within its elements – for any $\xi \in O(S)$, the unordered collection is denoted by $\tilde{\xi}$. For compact S, $O(S)$ may be canonically identified with $\sqcup_{n=0}^{\infty}\{x \in S^n : x_i \neq x_j, \forall 1 \leq i < j \leq n\}$. Finally, the space of totally-ordered marked locally-finite collection of points, with marks in K, will be denoted by $O(S, K)$.

For any $\gamma \in M(S)$ (or $O(S)$), we will use the notation $|\gamma|$ to denote the number of elements in γ, i.e., $|\gamma| = \gamma(S)$.

As mentioned in the introduction, the symbols $\mathtt{R}$ and $\mathtt{B}$ will be used to denote the types red and blue respectively. Moreover, we will let $\boldsymbol{C} = \{\mathtt{R}, \mathtt{B}\}$, and let a line over a color denote the opposite color, i.e., $\bar{\mathtt{R}} = \mathtt{B}$ and $\bar{\mathtt{B}} = \mathtt{R}$.

4.2 First-in-First-Match Matching Process on Compact Domains

In this section, we first give a formal definition of the process on a compact domain. Let D be a compact metric space with a Radon measure λ. The state of the process will contain information about the location, color and the order of arrival of the particles present in the system. Thus, the state space will be the set of totally-ordered collection of particles, with location in D and with marks in the set $\boldsymbol{C} = \{\mathtt{R}, \mathtt{B}\}$, namely $O(D, \boldsymbol{C})$. The order represents the order of arrival of particles into the system, and hence represents their priority when two particles are in contention to be matched to the same particle.

We will require the following notation to describe the evolution of the process. For a point $x \in D \times \boldsymbol{C}$, we denote the projection onto D by p_x and denote the projection onto $\boldsymbol{C}$ by c_x. For any point $x \in D \times \boldsymbol{C}$, we denote the set of incompatible points of opposite color by $N(x) := B(p_x, 1) \times \{\bar{c_x}\}$, where $B(z, r)$ denotes the ball of radius r centered at z. For any subset $A \subseteq D \times \boldsymbol{C}$, we set $N(A) := \cup_{x \in A} N(x)$. Let $\bar{\lambda} := \lambda \otimes m_c$, where m_c is the counting measure on $\boldsymbol{C}$. For any $\gamma \in O(D, \boldsymbol{C})$ and $x \in \gamma$, let γ^x be the element of $O(D, \boldsymbol{C})$ formed by $\{y \in \gamma : y <_\gamma x\}$ ordered as in γ. Further, if γ is represented as a list $(x_1, \ldots, x_n)$, then for any i, $1 \leq i \leq n$, we set

$\gamma_1^{i-1} := \gamma^{x_i} = (x_1, \ldots, x_{i-1})$. The region of the highest priority of a particle x in γ, denoted $W_{\gamma,x}$ (or just W_x if the context is clear), is defined to be the set $N(x)\backslash N(\gamma^x)$.

Let us recall the description of the dynamics in terms of the above notation. We consider a Markov jump process, $\{\eta_t\}_{t\in\mathbb{R}} \subset O(D, \boldsymbol{C})$. Suppose that a new particle, $y \in D \times \boldsymbol{C}$, arrives at time t. If the ordered collection $\eta_t \cap N(y)$ is non-empty, then the particle y *matches* to the lowest-ranked particle in this set and the matched particle is removed from η_t. Equivalently, y matches to $x \in \eta_t$ if and only if $y \in W_{\eta_t,x}$. Otherwise, y is added to the ordered set η_t at the end, so that $y > x$ for all $x \in \eta_t$, while the order among the elements of η_t is preserved. Additionally, independent of everything else, particles may depart on their own when they lose patience at rate $\mu > 0$.

This description fixes the form of the generator of the process, which is given by

$$\begin{aligned} Lf(\eta) := &\sum_{x\in\eta} \left(\mu + \bar{\lambda}(W_x)\right) [f(\eta\backslash x) - f(\eta)] \\ &+ \int_{D\times\boldsymbol{C}} \mathbb{1}(x \notin N(\eta))[f(\eta, x) - f(\eta)]\bar{\lambda}(dx), \end{aligned} \tag{4.1}$$

where f is a measurable function defined over $O(D, \boldsymbol{C})$. In the following we give an explicit construction of a process which will serve as the formal definition of our process. It can be easily verified that, if $\{\eta_t\}$ is the constructed process, then

$$\lim_{t\to 0+} \frac{1}{t} \operatorname{E}[f(\eta_t) - f(\eta_0)|\eta_0] = Lf(\eta_0),$$

for any bounded continuous function f over $O(D, \boldsymbol{C})$ (we refrain from identifying the full domain of the generator). For us, the form of the generator will be useful in characterizing the stationary distribution, while the explicit construction will be used later in the construction of the process on $\mathbb{R}^d$.

Let Φ be a Poisson point process on $D \times \mathbb{R}^+$, with i.i.d. marks in $\boldsymbol{C} \times \mathbb{R}^+$. The intensity of the point process is $2\lambda \otimes \ell$, where ℓ is the Lebesgue measure on $\mathbb{R}^+$. Both the marks are independent, with the color uniformly distributed and the other mark is an exponential random variable with parameter μ. Let $\eta_0 \in O(D, \boldsymbol{C})$ be the initial state of the system at time 0. Each point in $x \in \Phi$ is represented by four coordinates (p_x, b_x, c_x, w_x), with $p_x \in D$, $b_x, w_x \in \mathbb{R}^+$ and $c_x \in \boldsymbol{C}$. p_x denotes the spatial position of the point x, b_x denotes the time of its arrival, c_x denotes its color and w_x denotes its patience. The following display presents an algorithm for the construction of the process on compact domains.

- **Data:**
  - (a.) Φ: A realization of the arrivals.
  - (b.) η_0: A realization of the initial condition.
  - (c.) $t \in \mathbb{R}^+$: End time for simulation.
- **Result:** η_t: The final state of the system at time t.

1. Set $t_{old} = 0$.
2. For each $x \in \eta_0$, assign i.i.d. marks w_x, that are exponentially distributed with parameter μ.
3. Set $t_{new} = \min(\inf\{b_x : x \in \Phi_{(t_{old},\infty)}\}, \inf\{w_x : x \in \eta_{t_{old}}\})$. If $t_{new} > t$, quit and return $\eta_{t_{old}}$.
4. If t_{new} is due to arrival of a new particle (first infimum):
   - Let the particle be x.

- If there is a particle of opposite color in $B(p_x, 1)$:
  - Match to the first particle of opposite color in $\eta_{t_{old}} \cap B(p_x, 1)$ and remove that particle. This gives $\eta_{t_{new}}$.
- Else:
  - Add the particle to the end of $\eta_{t_{old}}$ to give $\eta_{t_{new}}$

5. Else if t_{new} is due to a particle $x \in \eta_{t_{old}}$ losing patience (second infimum), then remove this particle to yield $\eta_{t_{new}}$.
6. Set $t_{old} = t_{new}$. Go to Step 3.

4.2.1 Existence and Uniqueness of a Stationary Regime

In this section, we look at the stationary regime of the process on a compact domain D, defined in Section 4.2. We first show that there is a unique stationary measure for this process, and in the subsequent sections give a product form characterization.

It can be seen that if in the model, we have $\mu = 0$, the process $\{\eta_t\}$ does not have a stationary regime. Indeed, in this case, starting from $\eta_0 = \emptyset$, $|\eta_t| \geq |\Phi(D \times [0,t], \mathtt{R}) - \Phi(D \times [0,t], \mathtt{B})|$. The process on the right-hand side does not have a stationary regime. So, we need to assume $\mu > 0$. It is easy to argue then, using a standard coupling from the past or a Lyapunov technique, that there is a unique stationary distribution. For the sake of completeness, we give a coupling from the past construction of a stationary regime and prove that it is unique.

Suppose we have a bi-infinite time-ergodic Poisson point process Φ on $D \times \mathbb{R}$, with marks in $\boldsymbol{C} \times \mathbb{R}^+$, where the first coordinate is the color of the particle and the second coordinate is the time the particle is the patience, as in the construction in Section 4.2. We define the notion of a *regeneration time* of the Poisson point process Φ as follows. A time $t \in \mathbb{R}$ is called a regeneration time if for all $x \in \Phi$, with $b_x \leq t$, we have $t - b_x > w_x$. That is, there is no possibility that a particle arriving before t survives beyond time t. For any process, $\{\eta_s^r\}_{s \geq r}$, started with empty initial conditions at time r, and driven by the process Φ, we note that $\eta_s^r = \eta_s^t$, for all $s \geq t$ and all regeneration times $t \geq r$. Therefore, a stationary regime exists if we can show the existence of a sequence of regeneration times that diverge to $-\infty$ almost surely. This is an instance of a coupling from the past scheme. The following lemma provides such a sequence of regeneration times.

Lemma 4.2.1. *Under the setting of this section, there are infinitely many regeneration times in the list* $0, -1, -2, \ldots,$ *almost surely.*

Proof. Let us find the probability of the event, A_0, that 0 is a regeneration time. We have:

$$
\begin{aligned}
\mathrm{P}(A_0) &= \mathrm{P}(w_x < -b_x, \forall x \in \Phi, b_x < 0) \\
&= \mathrm{E} \prod_{x \in \Phi} \mathbb{1}(w_x < -b_x, b_x < 0) \\
&= \mathrm{E} \lim_{s \to \infty} e^{-s \int \mathbb{1}(w_x \geq -b_x) \Phi(dx)} \\
&\geq \limsup_{s \to \infty} \exp \left(\int_{D \times \boldsymbol{C}} \int_{\mathbb{R}^- \times \mathbb{R}^+} \left(e^{-s \mathbb{1}(w \geq -b)} - 1 \right) \mu e^{-\mu w} dw db \lambda(dp) \right) \\
&= \limsup_{s \to \infty} \exp \left(2\lambda(D) \int_{\mathbb{R}^+} (e^{-s} - 1) e^{-\mu b} db \right) \\
&= \limsup_{s \to \infty} \exp \left(2\lambda(D)(e^{-s} - 1)/\mu \right) \\
&= \exp\left(-2\lambda(D)/\mu\right) > 0,
\end{aligned}
$$

where in the third equation we have used the Fatou's lemma and the Laplace transform formula for Poisson point processes. Now, let A_n be event that $-n$ is a regeneration time. If θ_t is a time-shift operator, we have $A_n = \theta_{-n} A_0$. By time-ergodicity of Φ, A_n must occur infinitely often, almost surely. Thus, there are infinitely many regeneration times in the list $\{0, -1, -2, \ldots\}$. $\square$

The uniqueness of a stationary regime can also be show using a coupling argument. We only give an outline of this procedure here. Suppose we consider two stationary measures of the process. Let η_0^1 and η_0^2 be realizations of these two states. For large enough n, the probability that $|\eta_0^1| > n$ and $|\eta_0^2| > n$ is less than $\epsilon > 0$. Conditioned on this event we may couple the processes in a time T, using the coupling scheme from the previous lemma. Note that with such a scheme, we have $\mathrm{E}\,T < \infty$. Thus, the total variation distance $d_{TV}(\eta_t^1, \eta_t^2) \leq \epsilon + \mathrm{E}\,T/t$, using the coupling and the Markov inequalities. Since $d_{TV}(\eta_0^1, \eta_0^2) = d_{TV}(\eta_t^1, \eta_t^2)$, we must have that η_0^1 must be equal in distribution to η_0^2.

In the next section, we present a product form characterization of this steady state distribution. To do this, the key step is to construct the reversed process.

4.2.2 Product Form Characterization of the Steady State

Let Φ be the driving Poisson point process on $D \times \mathbb{R}$, with i.i.d marks in $\boldsymbol{C} \times \mathbb{R}^+$, that is given as data as defined in the coupling from the past construction in Section 4.2.1.

Lemma 4.2.1 implies that there exists a unique bi-infinite spatial matching process that is driven by Φ. We can thus define a (random) matching function, $m : \Phi \to D \times \mathbb{R} \times \boldsymbol{C}$, such that

$$m(x) = \begin{cases} (p_x, b_x + w_x, c_x) & \text{if } x \text{ exits on its own,} \\ (p_y, b_y, c_y) & \text{if } x \text{ matches to } y \in \Phi. \end{cases}$$

Conversely, m stores all the information necessary to build the process $\{\eta_t\}_{t\in\mathbb{R}}$. Indeed, the state of the spatial matching process we are interested in is given by

$$\eta_t = ((p_x, c_x) : x \in \Phi, b_x \leq t < b_{m(x)}),$$

where the list is ordered according to the birth-times, b_x.

To get a handle on the stationary distribution of this process, we shall create its reversed process. Taking inspiration from [1], we will include some additional data in the state of the system that will simplify the description of the reversed process. We shall consider a process that we call the *backward detailed process* generated by Φ and m. This process contains unmatched and matched particles in its state, and we distinguish these types by using marks "u" or "m" respectively. For any particle x in the state, s_x will refer to this mark.

For $t \in \mathbb{R}$, let

$$T_t := \min\{b_x : x \in \Phi, b_x \leq t < b_{m(x)}\}$$

be the time of arrival of the earliest among the unmatched particles at time t. Let

$$\Gamma_{\mathtt{u}} := \{(p_x, b_x, c_x, \mathtt{u}) : x \in \Phi, b_x \leq t < b_{m(x)}\}$$

be the set of (location, arrival-times and colors of) unmatched particles in $[T_t, t]$. Let

$$\begin{aligned}\Gamma_{\mathtt{m}} &:= \{(p_x, b_{m(x)}, c_x, \mathtt{m}) : x \in \Phi, b_x \leq t, T_t \leq b_{m(x)} \leq t\} \\ &= \{(p_{m(x)}, b_x, c_{m(x)}, \mathtt{m}) : x \in \Phi, b_{m(x)} \leq t, T_t \leq b_x \leq t, c_x \neq c_{m(x)}\} \\ &\qquad \cup \{(p_x, b_{m(x)}, c_x, \mathtt{m}) : x \in \Phi, b_x \leq t, T_t \leq b_{m(x)} \leq t, c_x = c_{m(x)}\},\end{aligned}$$

be the set of so-called *matched and exchanged* particles that are present in $[T_t, t]$. In the last expression, the first set of elements corresponds to particles that arrive in the relevant interval, $[T_t, t]$, and are matched by the time t; but instead of recording their positions and types, we record that of their matches. The second set of elements in that expression corresponds to particles arrive before t, that depart on their own in the time interval $[T_t, t]$; we record the time at which they depart.

Finally, define the backward detailed process, $\hat{\eta}_t$, be the list $((p_x, c_x, s_x) : x \in \Gamma_{\mathtt{u}} \cup \Gamma_{\mathtt{m}})$, ordered according to the values of $b_{(\cdot)}$. Clearly, the original process η_t can be obtained from $\hat{\eta}_t$ by removing the particles with marks $s_x = \mathtt{m}$. Notice that if $|\hat{\eta}_t| > 0$, the first element in $\hat{\eta}_t$, denoted by x_1, always satisfies $s_{x_1} = \mathtt{u}$.

The backward detailed process, $\hat{\eta}_t$, is a stationary version of a Markov process. A valid state of this Markov process is any finite list of elements, $(x_1, \ldots, x_n)$ from the set $D \times \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\}$ that satisfies the following definition.

Definition 4.2.1 (Definition of a valid state of $\hat{\eta}_t$)**.** We say that a finite list of elements $(x_1, \ldots, x_n)$, with $n \in \mathbb{N}$ and $x_i \in D \times \boldsymbol{C} \times \{\mathtt{m}, \mathtt{u}\}$, is a *valid* state of $\hat{\eta}_t$ if the following three conditions are satisfied:

1. $s_{x_1} = \mathtt{u}$, if $n \geq 1$.

2. For all $1 \leq i, j \leq n$, $s_{x_i} = s_{x_j} = \mathtt{u}$ and $d(p_{x_i}, p_{x_j}) \leq 1$ implies that $c_{x_i} = c_{x_j}$.

3. For all $1 \leq i < j \leq n$, $s_{x_i} = \mathtt{u}$, $s_{x_j} = \mathtt{m}$ and $d(p_{x_i}, p_{x_j}) \leq 1$ implies that $c_{x_i} = c_{x_j}$.

Condition 2 in the above definition essentially states that there cannot be a compatible unmatched pair in a valid state. This condition is equivalent to the condition that

$$\{y \in x_1, \ldots x_n : s_y = \mathtt{u}\} \cap N(\{y \in x_1, \ldots x_n : s_y = \mathtt{u}\}) = \emptyset.$$

Condition 3 cannot be violated, since otherwise the particle whose matched and exchanged pair is x_j could instead have matched to x_i that arrives earlier. This condition is equivalent to the condition that for all $1 \leq j \leq n$,

$$s_{x_j} = \mathtt{m} \implies x_j \notin N(\{y \in x_1, \ldots, x_j : s_y = \mathtt{u}\}.$$

Any valid state can be achieved by the process $\hat{\eta}_t$ in finite time with positive probability. Indeed, starting from the empty state, a valid state, $\hat{\eta}$, can result from empty state if the arrivals occur in the order listed in $\hat{\eta}$, with appropriate patience so that the particles in $\hat{\eta}$ marked $\mathtt{u}$ survive until time t, and the particles in $\hat{\eta}$ marked $\mathtt{m}$ exit on their own before the next arrival.

Transitions for $\hat{\eta}_t$ occur at the time of arrival of a new particle or at the event of a voluntary departure. At the time of a new arrival, we match and exchange the particles in the list $\hat{\eta}_t$, and at the time of a departure, we put the departing particle at the end of the list $\hat{\eta}_t$, while updating the mark to $\mathtt{m}$. Below, we describe the transitions and transition rates of this Markov process in detail.

The transitions and transition rates for $\hat{\eta}_t$ are as follows: Let $\hat{\eta} = (x_1, \ldots, x_n)$, $n \in \mathbb{N}$, be a valid state.

1. A particle $x_i \in \hat{\eta}$, with $s_{x_i} = \mathtt{u}$, loses patience: This occurs at rate μ. In this case, the new state is obtained by removing the x_i and inserting $(p_{x_i}, c_{x_i}, \mathtt{m})$ at the end of the list $\hat{\eta}$. Additionally, we need to prune leading matched and exchanged particles from $\hat{\eta}$ to obtain the new state.

2. A new particle $y = (p_y, c_y)$ arrives and is matched to a particle $x_i \in \hat{\eta}$, with $d(p_{x_i}, p_y) \leq 1$ and $c_{x_i} \neq c_y$: This occurs at rate $\bar{\lambda}(dy)\mathbb{1}(y \in W_{x_i})$. The new state is obtained by matching and exchanging the appropriate pair, and then pruning the leading matched and exchanged particles.

3. A new particle y arrives and there is no particle of opposite color within a distance 1 from it: This occurs at rate $\bar{\lambda}(dy)\mathbb{1}(y \notin N(\hat{\eta}))$. The new state is the one obtained by adding this new particle to the end of the list as an unmatched particle.

We now guess the time-reversed version of the backward-detailed process, and obtain its transition rates. The following construction will be useful in doing this. Consider a dual process $\check{\eta}_t$, that we call the *forward detailed process*. It is defined as follows: for $t \in \mathbb{R}$ let

$$Y_t := \max\{b_{m(x)} : x \in \Phi, b_x \leq t < b_{m(x)}\},$$

be the latest time at which all unmatched particles at time t are matched or exit. Let

$$\begin{aligned}\Xi_{\mathtt{m}} &:= \{(p_x, b_{m(x)}, c_x, \mathtt{m}) : x \in \Phi, b_x \leq t < b_{m(x)}\} \\ &= \{(p_{m(x)}, b_x, c_{m(x)}, \mathtt{m}) : x \in \Phi, b_{m(x)} \leq t < b_x, c_x \neq c_{m(x)}\} \\ &\qquad \cup \{(p_x, b_{m(x)}, c_x, \mathtt{m}) : x \in \Phi, b_x \leq t < b_{m(x)}, c_x = c_{m(x)}\},\end{aligned}$$

be the matched and exchanged particles corresponding to the particles that are born before time t, but have not been removed from the system by time t. Let

$$\Xi_{\mathtt{u}} := \{(p_x, c_x, b_x, \mathtt{u}) : x \in \Phi, t < b_x < Y_t, t < b_{m(x)}\},$$

be the particles in the relavant interval (t, Y_y), whose match arrives after time t. Now, let $\check{\eta}_t = ((p_x, c_x, s_x) : x \in \Xi_u \cup \Xi_{\mathtt{m}})$, ordered according to the values $b_{(\cdot)}$. Thus, the last element in the list $x_{|\check{\eta}|}$, always has $s_{x_{|\check{\eta}|}} = \mathtt{m}$. For motivations for these definitions, see [1].

Under our construction, using the bi-infinite Poisson point process Φ, the process $\{\check{\eta}_t\}_{t\in\mathbb{R}}$ is a stationary process. In fact, it is a stationary version of a Markov process, since all the arrivals and deaths are Markovian. The transitions and transition rates are defined in detail in Section A.4.

The underlying idea in obtaining the product form distribution is the following. For any list of elements γ, let $\mathrm{revx}(\gamma)$ be the list of elements in γ written in the reverse order, with the marks $\mathtt{u}$ and $\mathtt{m}$ flipped. Then we claim that, a version of time-reversal of the backward-detailed process $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$, is given by $\{\mathrm{revx}(\check{\eta}_t)\}_{t\in\mathbb{R}}$. That is,

$$\{\hat{\eta}_{-t}\}_{t\in\mathbb{R}} \stackrel{d}{=} \{\mathrm{revx}(\check{\eta}_t)\}_{t\in\mathbb{R}}.$$

Indeed, the two processes are exactly equal if $\check{\eta}_t$ is constructed using the time-reversal of Φ. We refrain from showing this observation in detail, and instead check the local balance conditions to obtain the product form result. See Appendix A.1 for definition of local balance conditions.

Before we state the main result of this section, we need to fix some notation. For any list $\gamma \in O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$ and $i \in \mathbb{N}$, we define $Q^i_{\mathtt{u}}(\gamma)$ to be the number of unmatched particles *among* the first i particles on γ, and define $Q^i_{\mathtt{m}}(\gamma)$ to be the number of matched particles *excluding* the first i particles of γ. In this notation, we may drop the reference to γ when the context is clear. Also, for the sake of brevity, we will write, for any $n \in \mathbb{N}$, $\rho(n) = 2\lambda(D) + n\mu$. We have the following result.

Theorem 4.2.2. *The density of the stationary measure of the backward detailed process, $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$, w.r.t. the measure $\oplus_{n=0}^{\infty}(\lambda \otimes m_c \otimes m_c)^n$ on $O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\}) = \sqcup_{n=0}^{\infty}(D \times \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})^n$, is given by*

$$\hat{\pi}(\gamma) = K\mathbb{1}(\gamma \text{ is valid}) \prod_{i=0}^{|\gamma|} \frac{1}{\rho(Q^i_{\mathtt{u}}(\gamma))} =: K\mathbb{1}(\gamma \text{ is valid})\hat{\Pi}(\gamma), \tag{4.2}$$

$$\hat{\pi}(\emptyset) = K, \tag{4.3}$$

where $\hat{\Pi}(\gamma) = \prod_{i=0}^{|\gamma|} \frac{1}{\rho(Q^i_{\mathtt{u}}(\gamma))}$, and where K is the normalizing constant,

$$K^{-1} = \sum_{n=0}^{\infty} \int_{(D\times\boldsymbol{C}\times\{\mathtt{u},\mathtt{m}\})^n} \mathbb{1}(\gamma \text{ is valid}) \prod_{i=0}^{n} \frac{1}{\rho(Q^i_{\mathtt{u}}(\gamma))} (\lambda \otimes m_c \otimes m_c)^{(n)}(d\gamma).$$

The proof of the above theorem is given in Appendix A.4.

For the stationary distribution π of the original process η_t, we compute the marginals of $\hat{\pi}$. Firstly, a state $\gamma = (x_1, \ldots, x_n) \in O(D, \boldsymbol{C})$ is a valid state of the process, $\{\eta_t\}_{t\in\mathbb{R}}$, if and only if $\{x_1, \ldots, x_n\} \cap N(\gamma) = \emptyset$.

Corollary 4.2.3. *The density of the stationary distribution π of the process η_t, w.r.t. the measure $\oplus_{n=0}^{\infty}(\lambda \otimes m_c)^n$ on $\sqcup_{n=0}^{\infty}(D \times \boldsymbol{C})^n$, is given by*

$$\pi(\gamma) = K\mathbb{1}(\gamma \textit{ is valid}) \prod_{i=1}^{|\gamma|} \frac{1}{\bar{\lambda}(N(\gamma_1^i)) + i\mu}, \tag{4.4}$$

$$\pi(\emptyset) = K, \tag{4.5}$$

where K is the same normalizing constant as in Theorem 4.2.2

Proof. We calculate the marginal distribution of the unmatched particles from the distribution in 4.2.2. Given that $\eta = (x_1, \ldots, x_n)$ is in the steady state, let l_i, $1 \leq i \leq n$, denote the number of matched particles present between x_i and x_{i+1} in the detailed version of the process. The only restriction that these particles must satisfy is that they must be incompatible with $x_1, \ldots x_i$. Integrating over the positions of each of the l_i particles gives a factor $\bar{\lambda}(D \times \boldsymbol{C} \backslash N(x_1, \ldots, x_i))$. Thus, we have:

$$\begin{aligned}
\pi(x_1, \ldots, x_n) &= K\mathbb{1}(x_1^n \text{ is valid}) \prod_{i=1}^{n} \sum_{l_i \in \mathbb{N}} \frac{(\bar{\lambda}(D \times \boldsymbol{C} \backslash N(x_1, \ldots, x_i)))^{l_i}}{(2\lambda(D) + i\mu)^{l_i+1}} \\
&= K\mathbb{1}(x_1^n \text{ is valid}) \prod_{i=1}^{n} \frac{1}{2\lambda(D) + i\mu - \bar{\lambda}(D \times \boldsymbol{C} \backslash N(x_1, \ldots, x_i))} \\
&= K\mathbb{1}(x_1^n \text{ is valid}) \prod_{i=1}^{n} \frac{1}{\bar{\lambda}(N(\eta_1^i)) + i\mu}.
\end{aligned}$$

□

4.2.3 Clustering Properties and the FKG Property

In this section, we focus on the stochastic geometric properties of the steady state arrangement of the particles in space D. Hence, we the order of the particles in the steady state. The Janossy density ([17]) of a point process intuitively is the relative probability of observing a given configuration of points with respect to a given reference measure. The Janossy density of the steady state distribution of our point process model, with respect to the Poisson point process on $D \times \boldsymbol{C}$ with intensity $\bar{\lambda}$ is given by dropping the order of particles in Equation 4.4. That is, the Janossy density is

$$\begin{aligned} \tilde{\pi}(x_1^n) &= K\mathbb{1}(x_1^n \text{ is valid})\tilde{\Pi}(x_1^n), \\ \tilde{\Pi}(x_1^n) &= \sum_{(X_1^n)\in\mathcal{P}(x_1^n)} \prod_{i=1}^{n} \frac{1}{\bar{\lambda}(N(X_1^i)) + i\mu}, \end{aligned} \tag{4.6}$$

where $\mathcal{P}(x_1^n)$ is the set of all permutations of $x_1, \ldots, x_n$, and K is a normalizing constant.

Let us take a moment to interpret the term $\bar{\lambda}(N(x_1^i))$ that appears in the above expression. This is the sum of the volumes of the union of balls around red particles in $x_1, \ldots, x_i$ and the union of balls around the blue particles in $x_1, \ldots, x_i$. Since such terms appear in the denominator in eq. 4.6, we expect that in the steady state the particles of the same color are clustered together.

In a variety of point processes, such as the one-particle Widom-Rowlinson model, or certain Cox processes [9], the FKG lattice property is a useful tool for proving stochastic dominance and clustering properties. In the case of Widom-Rowlinson model, the FKG inequality is also useful in showing the existence of a phase transition for the existence on the infinite domain [15].

The FKG lattice property defined on a measure ψ over a finite distributive lattice Ω states that for every $\xi, \gamma \in \Omega$,

$$\psi(\xi \vee \gamma)\psi(\xi \wedge \gamma) \geq \psi(\xi)\psi(\gamma). \tag{4.7}$$

If ψ satisfies eq. 4.7, it is said to be log-submodular. The FKG lattice property implies the positive association inequality:

$$\psi(fg) \geq \psi(f)\psi(g), \tag{4.8}$$

for all increasing functions f and g on Ω, where $\psi(f)$ represents the expectation of f with respect to ψ.

This theorem can also be extended to point processes in the continuum as follows (see [38, 15] for details). Let P is point process on a measurable space S, with Janossy density ψ with respect to a Poisson point process with intensity λ, the FKG lattice property states that:

$$\psi(\xi \cup \gamma)\psi(\xi \cap \gamma) \geq \psi(\xi)\psi(\gamma), \text{ forall } \xi, \gamma \in M(S). \tag{4.9}$$

Under this hypothesis, one can conclude positive association inequalities such as eq. 4.8, where f and g are now increasing functions on $M(S)$.

Remark 4.2.1. The FKG lattice property in the continuum point process case can also be stated in terms of the Papangelou conditional intensities: If $\varphi(x, \xi)$ is the Papangelou conditional intensity of a point process with Janossy density ψ, then eq. 4.9 is equivalent to $\varphi(x, \xi) \geq \varphi(x, \xi')$ for all $x \in S$ and $\xi, \xi' \in M(S)$ with $\xi \supseteq \xi'$ (see [38] for details).

In the following, we prove an FKG lattice property in the steady state version of our model, under a specific lattice structure defined on $M(D, \boldsymbol{C})$. Let $\xi = (\xi^{\mathtt{R}}, \xi^{\mathtt{B}})$ and $\gamma = (\gamma^{\mathtt{R}}, \gamma^{\mathtt{B}})$ be two configurations in $M(D, \boldsymbol{C})$, where $\xi^{\mathtt{R}}$ and $\gamma^{\mathtt{R}}$ are the red particles, and $\xi^{\mathtt{B}}$ and $\gamma^{\mathtt{B}}$ are the blue particles in these configurations. We say that $\xi > \gamma$ if and only if $\xi^{\mathtt{R}} \supset \gamma^{\mathtt{R}}$ and $\xi^{\mathtt{B}} \subset \gamma^{\mathtt{B}}$. We note that the FKG lattice property is satisfied in the binary particle Widom-Rowlinson model with the same lattice structure. In [15], the authors also use discretization based arguments to lift the positive associations result for this lattice structure in the continuum.

To prove the FKG lattice property in our setting, we need the following auxiliary lemma.

Lemma 4.2.4. *Let $(\alpha_i)_{i=1}^n$ and $(\beta_j)_{j=1}^m$ be two sets of positive numbers. Let $P(n,m)$ be the set of all increasing paths in the grid $[n] \times [m]$, so that for any $\sigma \in P(n,m)$, we have $\sigma(0) = (0,0)$, $\sigma(m+n) = (n,m)$, and $\sigma(i+1) - \sigma(i)$ is either $(1,0)$ or $(0,1)$, for all $0 \leq i < m+n$. Then, we have*

$$\sum_{\sigma \in P(n,m)} \prod_{i=1}^{n+m} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \prod_{i=1}^{n} \frac{1}{\alpha_i} \prod_{i=1}^{m} \frac{1}{\beta_i},$$

Here, σ_x and σ_y denote the x and y coordinate respectively.

The proof is by induction on m. The proof is not central to the current discussion, so we present it in Section A.5.

We are now ready to prove a weak form of FKG lattice property for the Janossy density given in eq. 4.6.

Theorem 4.2.5. *Let ξ and γ be disjoint finite subsets of $D \times \boldsymbol{C}$ such that the set $\xi \cup \gamma$ is valid. Then, we have*

$$\tilde{\pi}(\xi \cup \gamma)\tilde{\pi}(\emptyset) \geq \tilde{\pi}(\xi)\tilde{\pi}(\gamma). \tag{4.10}$$

Moreover, if ξ and γ are such that $N(\xi) \cap N(\gamma) = \emptyset$, then equality holds.

Proof. Let $|\xi| = n$ and $|\gamma| = m$. There is a canonical bijection between $\mathcal{P}(\xi \cup \gamma)$ and $\mathcal{P}(\xi) \times \mathcal{P}(\gamma) \times P(n, m)$. For $(\boldsymbol{a}, \boldsymbol{b}, \sigma) \in \mathcal{P}(\xi) \times \mathcal{P}(\gamma) \times P(n, m)$ we denote by $(\sigma, \boldsymbol{ab})$ the corresponding element in $\mathcal{P}(\xi \cup \gamma)$. We will use the coordinate-wise notation $\sigma = (\sigma_x, \sigma_y)$ for $\sigma \in \mathbb{Z}^2$. Also, for any sets $A, C \subset D \times \boldsymbol{C}$, let $N_C(A) = N(A \cap C)$.

We have

$$\begin{aligned}
\tilde{\Pi}(\xi \cup \gamma) &= \sum_{\substack{\boldsymbol{a} \in \mathcal{P}(\xi), \boldsymbol{b} \in \mathcal{P}(\gamma) \\ \sigma \in P(n,m)}} \prod_{l=1}^{n+m} \frac{1}{\bar{\lambda}(N((\sigma, \boldsymbol{ab})_1^l)) + l\mu} \\
&\geq \sum_{\substack{\boldsymbol{a} \in \mathcal{P}(\xi), \boldsymbol{b} \in \mathcal{P}(\gamma) \\ \sigma \in P(n,m)}} \prod_{l=1}^{n+m} \frac{1}{\bar{\lambda}(N_\xi((\sigma, \boldsymbol{ab})_1^l)) + \bar{\lambda}(N_\gamma((\sigma, \boldsymbol{ab})_1^l)) + l\mu} \\
&= \sum_{\substack{\boldsymbol{a} \in \mathcal{P}(\xi), \boldsymbol{b} \in \mathcal{P}(\gamma) \\ \sigma \in P(n,m)}} \prod_{l=1}^{n+m} \frac{1}{\bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(l)})) + \bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(l)})) + l\mu} \\
&= \tilde{\Pi}(\xi)\tilde{\Pi}(\gamma),
\end{aligned} \tag{4.11}$$

where in the second step we have used that

$$\bar{\lambda}(N((\sigma, \boldsymbol{ab})_1^l)) \leq \bar{\lambda}(N_\xi((\sigma, \boldsymbol{ab})_1^l)) + \bar{\lambda}(N_\gamma((\sigma, \boldsymbol{ab})_1^l)),$$

and in the last step we used Lemma 4.2.4 with $\alpha_i = \lambda(N(X_1^i)) + i\mu$ and $\beta_i = \lambda(N(Y_1^i)) + i\mu$. The proof now follows, since $K = \tilde{\pi}(\emptyset)$. Note also that if $N(\xi) \cap N(\gamma) = \emptyset$, then equality holds in eq. 4.11. $\square$

The above theorem is useful in the proof of the main result of this section only through the following corollary.

Corollary 4.2.6. *If $\gamma = \gamma^{\mathtt{R}} \cup \gamma^{\mathtt{B}}$ is a valid configuration (i.e., $\gamma \cap N(\gamma) = \emptyset$), where $\gamma^{\mathtt{R}}$ and $\gamma^{\mathtt{B}}$ are the red and blue particles respectively in γ. Then,*

$$\tilde{\pi}(\gamma) = \frac{1}{K}\tilde{\pi}(\gamma^{\mathtt{R}})\tilde{\pi}(\gamma^{\mathtt{B}}),$$

or equivalently,

$$\tilde{\Pi}(\gamma) = \tilde{\Pi}(\gamma^{\mathtt{R}})\tilde{\Pi}(\gamma^{\mathtt{B}}).$$

We now state the FKG lattice property for the usual subset ordering for the same type of particles.

Theorem 4.2.7. *Let ξ and γ be finite subsets of $D \times \{\mathtt{R}\}$. Then,*

$$\tilde{\Pi}(\xi \cup \gamma)\tilde{\Pi}(\xi \cap \gamma) \geq \tilde{\Pi}(\xi)\tilde{\Pi}(\gamma). \tag{4.12}$$

A similar property holds when ξ and γ are finite subsets of $D \times \{\mathtt{B}\}$.

The statement of the previous theorem is combinatorial in nature. However, its proof is interesting since we were able to use probabilistic tools by introducing artificial randomness. In particular, at one stage in the proof, we employ the FKG inequality on the lattice $\{0, 1\}^n$ (for some $n \in \mathbb{N}$). For the sake of exposition, this proof is moved to Appendix A.6.

Using Corollary 4.2.6 and Theorem 4.2.7, we conclude the main result of this section.

Corollary 4.2.8. *For any two finite subsets, $\xi = (\xi^{\mathtt{R}}, \xi^{\mathtt{B}})$ and $\gamma = (\gamma^{\mathtt{R}}, \gamma^{\mathtt{B}})$ of $D \times \boldsymbol{C}$, we have*

$$\tilde{\Pi}(\xi \vee \gamma)\tilde{\Pi}(\xi \wedge \gamma) \geq \tilde{\Pi}(\xi)\tilde{\Pi}(\gamma), \tag{4.13}$$

where the $\xi \vee \gamma = (\xi^{\mathtt{R}} \cup \gamma^{\mathtt{R}}, \xi^{\mathtt{B}} \cap \gamma^{\mathtt{B}})$ and $\xi \wedge \gamma = (\xi^{\mathtt{R}} \cap \gamma^{\mathtt{R}}, \xi^{\mathtt{B}} \cup \gamma^{\mathtt{B}})$.

Proof. By Corollary 4.2.6 we have

$$\tilde{\Pi}(\xi)\tilde{\Pi}(\gamma) = \tilde{\Pi}(\xi^{\mathtt{R}})\tilde{\Pi}(\xi^{\mathtt{B}})\tilde{\Pi}(\gamma^{\mathtt{R}})\tilde{\Pi}(\gamma^{\mathtt{B}}).$$

By Theorem 4.2.7, we have

$$\tilde{\Pi}(\xi^{\mathtt{R}})\tilde{\Pi}(\gamma^{\mathtt{R}})\tilde{\Pi}(\xi^{\mathtt{B}})\tilde{\Pi}(\gamma^{\mathtt{B}}) \leq \tilde{\Pi}(\xi^{\mathtt{R}} \cup \gamma^{\mathtt{R}})\tilde{\Pi}(\xi^{\mathtt{B}} \cap \gamma^{\mathtt{B}})\tilde{\Pi}(\xi^{\mathtt{R}} \cap \gamma^{\mathtt{R}})\tilde{\Pi}(\xi^{\mathtt{R}} \cup \gamma^{\mathtt{B}}).$$

Now, since $\xi \vee \gamma$ and $\xi \wedge \gamma$ are valid configurations, the proof follows by using Corollary 4.2.6 again. □

From Corollary 4.2.8 and the FKG inequality (see Appendix in [15]), we can conclude that the stationary measure is positively associated, i.e., for any two increasing functions f and g, then

$$\mathrm{E}_{\tilde{\eta}}\, f(\tilde{\eta})g(\tilde{\eta}) \geq \mathrm{E}_{\tilde{\eta}}\, f(\tilde{\eta})\, \mathrm{E}_{\tilde{\eta}}\, g(\tilde{\eta}), \tag{4.14}$$

where $\tilde{\eta}$ is a version of the unordered stationary process, and has the density $\tilde{\pi}$ with respect to the Poisson point process with intensity $\bar{\lambda}$. The above positive association inequalities also imply that the marginal point process of the red (or the blue points) are also positively associated. This can be seen by taking increasing functions f

and g that depend only on the red points (or the blue points). From this result and Corollary 3.1 of [9], we can conclude that $\tilde{\eta}^{\mathtt{R}}$ and $\tilde{\eta}^{\mathtt{B}}$ are weakly-super Poissonian. Intuitively, this means that the points are more clustered than the points in a Poisson point process of the same intensity.

4.2.4 Boundary Conditions and Monotonicity

In this section, we assume that D is a compact subset of the Euclidean domain $\mathbb{R}^d$, for some $d \geq 1$. We will use the FKG lattice property to prove monotonicity of measures under different boundary conditions. To state these theorems we will require the following notation. Let $\zeta \subset \mathbb{R}^d \backslash D \times \boldsymbol{C}$ be a valid state, i.e., $N(\zeta) \cap \zeta = \emptyset$. For any such boundary condition, we define a measure on $M(D, \boldsymbol{C})$ with Janossy density

$$\begin{aligned} \tilde{\pi}_{D,\zeta}(x_1^n) &= K_{D,\zeta} \mathbb{1}(N(x_1^n) \cap (\zeta \cup x_1^n) = \emptyset) \tilde{\Pi}_\zeta(x_1^n), \\ \tilde{\Pi}_\zeta(x_1^n) &= \sum_{(X_1^n) \in \mathcal{P}(x_1^n)} \prod_{i=1}^{n} \frac{1}{\bar{\lambda}(N(X_1^i) \cap N(\zeta)^c) + i\mu}. \end{aligned} \tag{4.15}$$

Three important boundary conditions are $\zeta = (\mathbb{R}^d \backslash D) \times \{\mathtt{R}\}$, $\zeta = \emptyset$ and $\zeta = (\mathbb{R}^d \backslash D) \times \{\mathtt{B}\}$. These are termed the *red*, the *free* and the *blue* boundary conditions respectively. We use special notation for the densities with these boundary conditions, namely, $\tilde{\pi}_{S,\mathtt{R}}$, $\tilde{\pi}_S$ and $\tilde{\pi}_{S,\mathtt{B}}$.

The boundary conditions can also be partially ordered: let $\zeta_1 \geq \zeta_2$ if and only if $\zeta_1^{\mathtt{R}} \supset \zeta_2^{\mathtt{R}}$ and $\zeta_1^{\mathtt{B}} \subset \zeta_2^{\mathtt{B}}$. We are now in a position to state the first result of this section.

Theorem 4.2.9. *Let $D \subset \mathbb{R}^d$ be compact set. Let $\zeta_1 \geq \zeta_2$ be two boundary conditions on D. Then, the measure with density $\tilde{\pi}_{D,\zeta_1}$ stochastically dominates the measure* *with density* $\tilde{\pi}_{D,\zeta_2}$.

Outline of the proof. By Holley's inequality [41], it is enough to prove that for two states, η and $\gamma \in M(D, \boldsymbol{C})$, we have

$$\tilde{\pi}_{D,\zeta_1}(\eta \vee \gamma)\tilde{\pi}_{D,\zeta_2}(\eta \wedge \gamma) \geq \tilde{\pi}_{D,\zeta_1}(\eta)\tilde{\pi}_{D,\zeta_2}(\gamma). \tag{4.16}$$

We first note that if $N(\eta) \cap (\zeta \cup \eta) = \emptyset$ and $N(\gamma) \cap (\zeta_2 \cup \gamma) = \emptyset$, then

$$N(\eta \vee \gamma) \cap (\zeta_1 \cup (\eta \vee \gamma)) = N(\eta \wedge \gamma) \cap (\zeta_2 \cup ((\eta \wedge \gamma)) = \emptyset.$$

Since the red and blue subsets do not interact by Corollary 4.2.6, we only need to show that if $\zeta_2 \subset \zeta_1 \subset (\mathbb{R}^d \backslash D) \times \{\mathtt{R}\}$ and η and $\gamma \in M(D, \{\mathtt{R}\})$, then

$$\tilde{\Pi}_{\zeta_1}(\eta \cup \gamma)\tilde{\Pi}_{\zeta_2}(\eta \cap \gamma) \geq \tilde{\Pi}_{\zeta_1}(\eta)\tilde{\Pi}_{\zeta_2}(\gamma).$$

The proof of the last statement follows by simple modification of the proof of Theorem 4.2.7, presented in Appendix A.6. Specifically, Equation A.20 in the appendix is modified to

$$\begin{aligned}
&\mathrm{E} \prod_{i=1}^{n+m+2k} \left(\begin{matrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\zeta_1)^c) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\zeta_2)^c) + \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}w(i)}) \cap N(\zeta_2)^c) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_2)^c) + i\mu \end{matrix} \right)^{-1} \\
&\geq \\
&\mathrm{E} \prod_{i=1}^{n+m+2k} \left(\begin{matrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\zeta_1)^c) + \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\zeta_2)^c) + \bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) + \bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}w(i)}) \cap N(\zeta_2)^c) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}x(i)}) \cap N(\mathfrak{c}_1^{\mathfrak{S}z(i)}) \cap N(\zeta_1)^c) - \bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}y(i)}) \cap N(\bar{\mathfrak{c}}_1^{\mathfrak{S}w(i)}) \cap N(\zeta_2)^c) + i\mu \end{matrix} \right)^{-1},
\end{aligned}$$

where the expectation is over a uniformly random choice

$$\begin{aligned}
&(\mathfrak{S}, \mathfrak{a}\mathfrak{b}\mathfrak{c}\bar{\mathfrak{c}}) \in P(n, m, k, k) \times \mathcal{P}(\eta \backslash \gamma) \times \mathcal{P}(\gamma \backslash \eta) \times \mathcal{P}(\eta \cap \gamma) \times \mathcal{P}(\eta \cap \gamma), \\
&n = |\eta \backslash \gamma|,\ m = |\gamma \backslash \eta|,\ k = |\eta \cap \gamma|.
\end{aligned}$$

The rest of the proof follows similar steps to the proof in Appendix A.6. $\square$

Remark 4.2.2. The above proof can be suitably modified to give the following interesting result. Let $D_1 \subset D_2$ be two compact subsets of $\mathbb{R}^d$. With an abuse of notation, let $\tilde{\pi}_{D_2,\mathtt{R}}(\eta)$ denote the marginal-Janossy density of observing η in D_1, under the measure with density $\tilde{\pi}_{D_2,\mathtt{R}}$. Similarly, we overload the notation for $\tilde{\pi}_{D_2,\mathtt{B}}$. With this notation, we may prove that $\tilde{\pi}_{D_1,\mathtt{R}} \geq \tilde{\pi}_{D_2,\mathtt{R}}$ and $\tilde{\pi}_{D_1,\mathtt{B}} \leq \tilde{\pi}_{D_2,\mathtt{B}}$. For the proof, we apply Holley's inequality, which requires that the following inequality holds:

$$\tilde{\pi}_{D_1,\mathtt{R}}(\eta \vee \gamma)\tilde{\pi}_{D_2,\mathtt{R}}(\eta \wedge \gamma) \geq \tilde{\pi}_{D_1,\mathtt{R}}(\eta)\tilde{\pi}_{D_2,\mathtt{R}}(\gamma),$$

where $\eta, \gamma \in M(D_1, \boldsymbol{C})$ are two valid configurations. This is follows from the inequality

$$\tilde{\pi}_{D_1,\mathtt{R}}(\eta \vee \gamma)\tilde{\pi}_{D_2,\mathtt{R}}(\xi \cup (\eta \wedge \gamma)) \geq \tilde{\pi}_{D_1,\mathtt{R}}(\eta)\tilde{\pi}_{D_2,\mathtt{R}}(\xi \cup \gamma),$$

where $\xi \in M(D_1 \backslash D_2, \boldsymbol{C})$ is any configuration such that $\xi \cup \gamma$ is a valid configuration. The later inequality can be proved using similar ideas used in the proof of Theorem 4.2.7 in Appendix A.6. We skip the details of the cumbersome calculations here. We will only remark here that, such a monotonicity of measures allows us to consider the limiting extremal measures $\lim_{D_n \nearrow \mathbb{R}^d} \tilde{\pi}_{D_n,\mathtt{R}}$ and $\lim_{D_n \nearrow \mathbb{R}^d} \tilde{\pi}_{D_n,\mathtt{B}}$, on the infinite Euclidean domain $\mathbb{R}^d$. In the next few sections, we will consider the First-in-first-match process on infinite Euclidean domains, and prove the existence of a stationary regime. We leave the job of exploring of the connection between the stationary measure so obtained and these limiting measures to future work.

4.3 First-in-First-Match Matching Process on Euclidean domains

In the following few sections, we extend the definition of the process previously given on a compact space to a non-compact space. We will specifically focus on the Euclidean space $\mathbb{R}^d$, for some $d \geq 1$. The following methodology can be extended to other non-compact spaces that satisfy certain additional assumptions, but we refrain from presenting these results in complete generality. When $D = \mathbb{R}^d$, there are infinitely many arrival and departure events that are triggered in any finite interval of time. So, the process cannot be constructed as a jump Markov process using the algorithm presented in Section 4.2.

The key to the definition and construction of the process on $\mathbb{R}^d$ is the following viewpoint. Let us look understand this viewpoint the bounded setting first, and see its relation to the algorithm given in Section 4.2. Let D be a bounded space. Let $\Phi \in M(D \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$ and $\eta_0 \in O(D, \boldsymbol{C} \times \mathbb{R}^+)$ be the driving Poisson point process and the initial condition, respectively, as defined in the Section 4.2. Here, each particle $x \in \eta_0$ is of the form $x = (p_x, c_x, w_x)$, where p_x, c_x and w_x are the position, color and patience of the particle. Similarly, any point $x \in \Phi$ is of the form $x = (p_x, b_x, c_x, w_x)$, where additionally b_x denotes the arrival time in Φ.

We will treat Φ as an element of $O(D \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$, where the points of Φ are ordered according to their birth times, as in Section 4.2. We will also treat η_0 as an element of $O(D \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$, by setting $b_x = 0$ for all $x \in \eta_0$, while preserving the order in η_0. Moreover, we also consider $\Phi \cup \eta_0$ as an element of $O(D \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$, where all the elements of η_0 are ranked less than the elements of Φ, while preserving the order within these sets.

We define a function $\kappa : \Phi \cup \eta_0 \to D \times \boldsymbol{C} \times \mathbb{R}^+ \sqcup \{\diamondsuit\}$, which we call the *killing* function, that is created as the process is built by the algorithm in Section 4.2. We set $\kappa(x)$ according to the following exhaustive set of rules.

1. If x arrives after y and matches with it, then $\kappa(x) = y$.
2. If x is accepted into the system, then $\kappa(x) = \diamondsuit$.
3. If $x \in \eta_0$, then $\kappa(x) = \diamondsuit$.

According to the description of the process, the function κ satisfies the following recursive property. For any $x \in \Phi$,

$$\kappa(x) = \min \left\{ y \in (\Phi \cup \eta_0) : \substack{y<x,\ c_y \neq c_x,\ d(p_y, p_x)<1,\ \kappa(y)=\diamondsuit,\ b_y + w_y > b_x, \\ (\forall z,\ y<z<x,\ d(p_y,p_z)<1,\ c_z = c_x,\ \kappa(z) \neq y)} \right\}, \tag{4.17}$$

where the minimum above is set equal to $\diamondsuit$ if the above set is empty. In words, the conditions in the definition of the above set select particles of opposite color that arrive before (or are ranked lower), are accepted when they arrive, whose patience does not run out before x arrives, and are not matched to any particle arriving before x.

The above recursive property can serve as a definition of the function κ, even in the non-compact case, if we can show that the recursive definition terminates almost surely. We note that we could compute the value of $\kappa(x)$ if we knew all values of κ on points in $\Phi \cup \eta_0$ that are within a spatial distance 2 from x and that arrive before it. It is also enough to just know all the values of κ for point in Φ within a spatial distance 4 that arrive before x. The following lemma provides the tool needed to claim the termination of the recursive definition.

Lemma 4.3.1. *Suppose $\Phi \in O(\mathbb{R}^d \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$ be a Poisson point process of intensity $\ell \otimes \ell \otimes m_c \otimes \mu e^{-\mu x}\ell(dx)$, where ℓ is the Lebesgue measure on corresponding Euclidean spaces. Then, there is no infinite sequence $\{y_n\}_{n\in\mathbb{N}} \subset \Phi$ such that $b_{y_i} > b_{y_{i+1}}$ and $d(p_{y_i}, p_{y_{i+1}}) \leq 4$ for all $i > 0$.*

The proof of this lemma is similar the same as the proof of Lemma 3.2.1 of Chapter 3.

The above lemma ensures that we can obtain the value of $\kappa(x)$ for any $x \in \Phi$ by recursively applying the Equation 4.17. The process in turn can be defined by setting

$$\eta_t = \{(p_y, c_y) : y \in \Phi \cup \eta_0,\ \kappa(y) = \diamondsuit,\ b_y \leq t < b_y + w_y, \text{ and } \kappa(z) = y \implies t < b_z\},$$

for all $t > 0$. On the unbounded domain $\mathbb{R}^d$, this will serve as the definition of the FIFM spatial matching process.

4.3.1 Construction of Stationary Regime on Euclidean domains

The simple coupling form the past argument presented in Section 4.2 does not pass in the case where $\lambda(D) = \infty$, since we cannot find a sequence of regeneration times going to ∞ in this case. We can show however that a coupling from the past argument can still be performed locally in space. This is done by first showing that, for a compact subset C of the domain $\mathbb{R}^d$, there exists time T_C beyond which two simulations agree for all times t beyond time T_C (So, T_C is not a stopping time). A key ingredient in proving the existence of T_C is an analysis of the decay in first order moment measures of the discrepancies between the two point patterns in simulation. Using this analysis, we are also able to bound the moments of T_C, which enables the application of ergodic theorems, as applied in the simple coupling from the past construction in Section 4.2. In the following sections, we present this coupling from the past argument in detail.

4.3.2 A Coupling of Two Processes

We will first obtain some results about a coupling of two different processes, $\{\eta_t^1\}_{t\geq 0}$ and $\{\eta_t^2\}_{t\geq 0}$, starting from the two different initial conditions, but driven by the same driving process $\Phi \in O(\mathbb{R}^d \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$. Let η_0^1 and η_0^2 be the two valid initial conditions ($\eta_0^i \cap N(\eta_0^i) = \emptyset$) that are spatially stationary. At any time $t \geq 0$, there are some particles that are present in both processes. These particles will be called *Regular* particles, and denoted by R_t. Call those particles that are present in η_t^1 and absent in η_t^2 as *Zombies*, and those that are absent in η_t^1 and present in η_t^2 as *Antizombies*. We denote them by Z_t and A_t respectively. Further, call particles in $Z_t \cup A_t$ as *Special* particles, and denote them by S_t.

We now prove that the density of the special particles decays exponentially to zero.

Theorem 4.3.2. *There exist constants $c > 0$, $\beta_{S_t} < \beta_{S_0} e^{-ct}$, for all $t > 0$, where β_{S_t} is the intensity of the special points, S_t.*

Proof. Let $K \subset \mathbb{R}^d$ be compact. Define $K^+ = \{y \in \mathbb{R}^d : d(y, K) \leq 1\}$ and $K^- = \{y \in \mathbb{R}^d : d(y, K^c) \leq 1\}^c$, $\partial K^+ = K^+ - K$ and $\partial K^- = K - K^-$. Also, let for any $T \subset \mathbb{R}^d$, $T_{\boldsymbol{C}}$ denote the set $T \times \boldsymbol{C}$. Now, we will compute the difference $\mathrm{E}[Z_{t+\delta}(K_{\boldsymbol{C}}) - Z_t(K_{\boldsymbol{C}})]$ for small $\delta > 0$, by tracking the changes that may occur in the short time interval $(t, t+\delta)$. Recall that we use the notation W_x^i to denote the domain of influence of the particle $x \in \eta_t^i$, $i = 1, 2$. Also, in the following we have $\bar{\lambda} := \ell \otimes m_c$ on $\mathbb{R}^d \times \boldsymbol{C}$. The following possibilities may occur:

- A zombie in K exits on its own by losing patience. The expected difference is

$$-\mu\delta \, \mathrm{E}\, Z_t(K_{\boldsymbol{C}}) + o(\delta). \tag{4.18}$$

- With probability $o(\delta)$, two or more particles arrive or depart in K^+. The expected change in $Z_t(K)$, given that this occurs, is $o(\delta)$.

- A zombie in K matches with a particle arriving in K^c, which is accepted in the process η_t^2. This results in a difference of

$$-\delta \, \mathrm{E} \sum_{x \in Z_t \cap K_{\boldsymbol{C}}} \bar{\lambda}(W_x^1 \cap K_{\boldsymbol{C}}^c \cap (N(\eta_t^2))^c) + o(\delta). \tag{4.19}$$

- A zombie in K matches with a particle arriving in K, which is accepted in the process η_t^2. This new particle is an antizombie. The resulting change is

$$-\delta \, \mathrm{E} \sum_{x \in Z_t \cap K_{\boldsymbol{C}}} \bar{\lambda}(W_x^1 \cap K_{\boldsymbol{C}} \cap (N(\eta_t^2))^c) + o(\delta). \tag{4.20}$$

- A zombie in K matches with an arriving particle, which also matches with some particle in K^c in the complementary process. This results in an expected change of

$$-\delta\,\mathrm{E}\sum_{x\in Z_t\cap K_{\boldsymbol{C}}}\sum_{y\in\eta_t^2\cap K_{\boldsymbol{C}}^c}\bar{\lambda}(W_x^1\cap W_y^2)+o(\delta). \tag{4.21}$$

- A zombie in K matches with an arriving particle, which also matches with some anti-zombie in K. This results in an expected change of

$$-\delta\,\mathrm{E}\sum_{x\in Z_t\cap K_{\boldsymbol{C}}}\sum_{y\in A_t\cap K_{\boldsymbol{C}}}\bar{\lambda}(W_x^1\cap W_y^2)+o(\delta). \tag{4.22}$$

- An anti-zombie matches with a particle arriving in K, that is accepted in the complementary process. This particle becomes a zombie. This results in an expected change of

$$\delta\,\mathrm{E}\sum_{x\in A_t}\bar{\lambda}(W_x^2\cap K_{\boldsymbol{C}}\cap(N(\eta_t^1))^c)+o(\delta). \tag{4.23}$$

- An arriving particle matches with a zombie in K^c and a regular particle in $\eta_t^2\cap K_{\boldsymbol{C}}$. The regular particle turns into a zombie. This results in a change of

$$\delta\,\mathrm{E}\sum_{\substack{x\in R_t\cap K_{\boldsymbol{C}}\\ y\in Z_t\cap K_{\boldsymbol{C}}^c}}\bar{\lambda}(W_x^2\cap W_y^1)+o(\delta) \tag{4.24}$$

Hence, we have:

$$\begin{aligned}
&\mathrm{E}[Z_{t+\delta}(K_{\boldsymbol{C}}))-Z_t(K_{\boldsymbol{C}})]\\
&=-\mu\delta\,\mathrm{E}\,Z_t(K_{\boldsymbol{C}})+\delta\,\mathrm{E}\Bigg[-\sum_{x\in Z_t\cap K_{\boldsymbol{C}}}\bigg(\bar{\lambda}(W_x^1\cap K_{\boldsymbol{C}}^c\cap(N(\eta_t^2)))\\
&\qquad-\bar{\lambda}(W_x^1\cap K_{\boldsymbol{C}}\cap(N(\eta_t^2))^c-\sum_{y\in\eta_t^2\cap K_{\boldsymbol{C}}^c}\bar{\lambda}(W_x^1\cap W_y^2)-\sum_{y\in A_t\cap K_{\boldsymbol{C}}}\bar{\lambda}(W_x^1\cap W_y^2)\bigg)\\
&\quad+\sum_{x\in A_t}\bar{\lambda}(W_x^2\cap K_{\boldsymbol{C}}\cap(N(\eta_t^1))^c)+\sum_{\substack{x\in R_t\cap K_{\boldsymbol{C}}\\ y\in Z_t\cap K_{\boldsymbol{C}}^c}}\lambda(W_x^2\cap W_y^1)\Bigg]+o(\delta).
\end{aligned}$$

Taking only the 2nd, 5th and 6th terms of the square braces of the above expression, dividing by δ, and taking the limit as $\delta \to 0$, we obtain:

$$\frac{d\,\mathrm{E}\,Z_t(K_{\boldsymbol{C}})}{dt} \le -\,\mu\,\mathrm{E}\,Z_t(K_{\boldsymbol{C}}) + \mathrm{E}\left[-\sum_{x\in Z_t\cap K_{\boldsymbol{C}}} -\lambda(W_x^1 \cap K_{\boldsymbol{C}} \cap (N(\eta_t^2))^c\right.$$
$$\left. + \sum_{x\in A_t} \lambda(W_x^2 \cap K_{\boldsymbol{C}} \cap (N(\eta_t^1))^c) + \sum_{\substack{x\in R_t\cap T_{\boldsymbol{C}}\\ y\in Z_t\cap T_{\boldsymbol{C}}^c}} \lambda(W_x^2 \cap W_y^1)\right].$$

We have similar bounds for derivatives of $\mathrm{E}\,A_t(K_{\boldsymbol{C}})$. Adding these expressions we obtain:

$$\begin{aligned}\frac{d}{dt}\,\mathrm{E}\,S_t(K_{\boldsymbol{C}}) &\le -\mu\,\mathrm{E}\,S_t(K_{\boldsymbol{C}}) + \mathrm{E}\left[2\sum_{\substack{x\in R_t\cap K_{\boldsymbol{C}}\\ y\in Z_t\cap K_{\boldsymbol{C}}^c}} \bar{\lambda}(W_x^2 \cap W_y^1) + 2\sum_{\substack{x\in R_t\cap K_{\boldsymbol{C}}\\ y\in A_t\cap K_{\boldsymbol{C}}^c}} \bar{\lambda}(W_x^2 \cap W_y^1)\right]\\ &\le -\mu\,\mathrm{E}\,S_t(K_{\boldsymbol{C}}) + 4\bar{\lambda}(\partial K_{\boldsymbol{C}}^+ \cup \partial K_{\boldsymbol{C}}^-).\end{aligned}$$

Taking the limit $K \nearrow \mathbb{R}^d$, by spatial ergodicity of the process S_t, we obtain

$$\frac{d}{dt}\beta_{S_t} \le -\mu\beta_{S_t}, \tag{4.25}$$

from which we can conclude that $\beta_{S_t} \le \beta_{S_0} e^{-\mu t}$. □

4.3.3 The Coupling from the Past Construction

In this section, we present the coupling from the past construction of the stationary regime. Let Φ be a doubly infinite Poisson point process, as in Lemma 4.2.1. That is, Φ is a Poisson point process defined on $\mathbb{R}^d \times \mathbb{R}$, with i.i.d. marks in $\boldsymbol{C} \times \mathbb{R}^+$. The intensity of the point process is $2\ell \otimes \ell$ and the marks are independent of each other, with the color uniformly distributed in $\boldsymbol{C}$ and the patience an exponential random variable. Let $\{\theta_t\}_{t\in\mathbb{R}}$ be a set of time-shift operators such that $\Phi\circ\theta_t(L\times A) = \Phi(L\times(A-t))$. Let $\{\eta_t^T\}_{t\geq -T}$, $T\in\mathbb{N}$, be a sequence of processes starting at time $-T$ with empty initial conditions and driven by arrivals from Φ. We have $\eta_t^T = \eta_{t+T}^0\circ\theta_{-T}$.

The processes η_t^1 and η_t^0 are driven by the same Poisson point process Φ beyond time 0. Treating the particles in η_0^1 as the initial conditions, we have a coupling of $\{\eta_t^1\}_{t\geq 0}$ and $\{\eta_t^0\}_{t\geq 0}$ as in Section 4.3.2. The discrepancies on any bounded set goes to zero exponentially fast by Theorem 4.3.2. In the following lemma, we show that such an exponential rate of convergence is enough to show that the time after which discrepancies never appear in any compact region has finite expectation. For any compact $K\subset D$, define

$$\tau^0(K) := \inf\{t>0 : \eta_s^1|_K = \eta_s^0|_K,\ s\geq t\}, \tag{4.26}$$

and in the following, let S_t denote the set of discrepancies, $\eta_t^0\triangle\eta_t^1$. Note that $\tau^0(K)$ is not a stopping time in our setting, since, first, $S_t\neq\emptyset$ for all $t\geq 0$ a.s. (there are always discrepancies somewhere in $\mathbb{R}^d$) by spatial ergodicity, and second, once discrepancies vanish in K, they can reappear due to interactions with the particles from outside of K.

We have

Lemma 4.3.3. *For all compact $K\subset\mathbb{R}^d$, $\mathrm{E}\,\tau^0(K)<\infty$.*

Proof. We view $S_t(K)$, $t\geq 0$, as a birth-death process. Let

$$S_t(K) = S_0(K) + S^+(0,t] - S^-(0,t],$$

where S^+ and S^- are simple counting processes. Since new special particles only result from interaction of arriving particles with existing special particles, the rate of increase in S^+ is bounded above by

$$\sum_{x \in S_t \cap K} \ell(B(x,1)) = \ell(B(0,1)) S_t(K).$$

Hence,

$$\mathrm{E}\, S^+[0,\infty) \leq \ell(B(0,1)) \int_0^\infty \mathrm{E}\, S_t(K) dt < \infty.$$

Since total departures are less than total arrivals,

$$\mathrm{E}\, S^-[0,\infty) \leq \mathrm{E}\, S_0(K) + \mathrm{E}\, S^+[0,\infty).$$

This in particular shows that $S^+[0,\infty)$ and $S^-[0,\infty)$ exist and are finite a.s. Thus, $\lim_{t\to\infty} S_t(K)$ also exists and is finite a.s. By dominated convergence theorem, $\lim_{t\to\infty} \mathrm{E}\, S_t(K) = \mathrm{E} \lim_{t\to\infty} S_t(K)$. Thus, by Theorem 4.3.2, $\lim_{t\to\infty} S_t(K) = 0$, a.s. This shows that $\tau(K) < \infty$ a.s.

Further, we have:

$$\begin{aligned}
\mathrm{E}\, \tau(K) &\leq \mathrm{E} \int_0^\infty t S^-(dt) \\
&= \mathrm{E} \int_0^\infty t S^+(dt) - \mathrm{E} \int_0^\infty t S(dt) \\
&= \mathrm{E} \int_0^\infty t S^+(dt) + E \int_0^\infty S(t) dt \\
&\leq \ell(B(0,1)) \int_0^\infty t\, \mathrm{E}\, S_t(K) dt + \int_0^\infty \mathrm{E}\, S_t(K) dt \\
&< \infty.
\end{aligned}$$

$\square$

Now, let $\tau^T(K)$ be defined as

$$\tau^T(K) := \inf\{t > -T : \eta_s^{T+1}|_K = \eta_s^T|_K,\ s \geq t\}.$$

$\tau^T(K)$ denotes the time at which executions of processes $\{\eta_t^T\}$ and $\{\eta_t^{T+1}\}$ coincide inside the set K. We have

$$\tau^T(K) = \tau^0(K) \circ \theta_{-T} - T.$$

That is,

$$\tau^T(K) + T = \tau^0(K) \circ \theta_{-T}. \tag{4.27}$$

Therefore, the sequence $\tau^T(K)+T$ is a stationary and ergodic sequence. By Birkhoff's point-wise ergodic theorem and by Lemma 4.3.3,

$$\lim_{T\to\infty} \sum_{i=0}^{T} \frac{\tau^i(K) + i}{T} = \mathrm{E}\, \tau^0(K) < \infty, \text{ a.s.}$$

Therefore the last term in the summation, $\frac{\tau^T(K)+T}{T}$ goes to 0 as $T \to \infty$. From this we conclude that

$$\lim_{T\to\infty} \tau^T(K) = -\infty. \tag{4.28}$$

This result has the following implication. For every realization of Φ, any compact set K and $t \in \mathbb{R}$, there exists a $k \in \mathbb{N}$ such that for all $T > k$, $\tau^0(K) \circ \theta_{-T} - T < t$. That is, the execution of all processes $\{\eta_s^T\}$, $T > k$, coincides at time t on the compact set K. Then, locally in the total variation sense, the following limit is well-defined a.s. on the same probability space:

$$\eta_t := \lim_{T\to\infty} \eta_t^T. \tag{4.29}$$

The process η is $\{\theta_n\}_{n\in\mathbb{Z}}$ compatible since

$$\begin{aligned}\eta_t \circ \theta_1 &= \lim_{T\to\infty} \eta_t^T \circ \theta_1 \\ &= \lim_{T\to\infty} \eta_{T+t}^0 \circ \theta_{-T+1} \\ &= \lim_{T\to\infty} \eta_{t+1+T-1}^0 \circ \theta_{-T+1} \\ &= \eta_{t+1}.\end{aligned}$$

Further, the process can also be shown to be $\{\theta_s\}_{s\in\mathbb{R}}$ compatible. Indeed, fix $s \in \mathbb{R}$. Let us implement a similar coupling from the past procedure, but with processes η^{T+s} that start with empty initial conditions at time $-T-s$, $T \in \mathbb{N}$. If $\hat{\eta}$ is the process obtained in such as manner, it can be shown that $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$ is equal to $\{\eta_t\}_{t\in\mathbb{R}}$ generated as above. Thus, $\eta_{t+s} = \eta_t \circ \theta_s$. This proves the $\{\eta_t\}_{t\in\mathbb{R}}$ is the stationary regime of the process.

4.4 Concluding Remarks and Future Work

In this chapter, we focused on a dynamic matching model with a natural policy, under the added assumption that particles may depart without being matched. We were able to find a characterization of the steady state distribution of the particles. Then using this characterization, we proved the FKG lattice property, which in turn enabled us to conclude that the property that particles of the same type are weakly-super Poissonian. We also prove that there is a stationary regime for the dynamics is the infinite Euclidean domain, $\mathbb{R}^d$.

The two particle Widom-Rowlinson model is a simpler model, where the FKG property, as satisfied by our model, is also satisfied. There, this property is used to show the existence of Markov random fields on the infinite Euclidean domain, $\mathbb{R}^d$. In the future, we would like to see whether this construction works in our setting, and how it relates to the stationary regime constructed on the domain $\mathbb{R}^d$.

The gray version of two particle WR model, which is obtained by removing the reference to the colors of the points also satisfies an FKG inequality – we are unable to prove this in our setting. This is a fundamental step in the symmetry breaking argument of [15]. We have not found such an argument in our setting. A symmetry breaking argument will show, for certain values of the parameter, that there are more red points than blue points in the steady state, or vice-versa. This also has implications on the relaxation times of the Markov process on finite domains. In the future, we would like to explore these problems.

Chapter 5

A Large Deviation Principle for Poisson Point Processes and Shot-noise Fields

5.1 Introduction

In this chapter, we study a Large deviation principle for dense Poisson point processes on a compact subset of $\mathbb{R}^d$. We then consider its application to obtain an LDP for its shot-noise fields. This problem is motivated by the need to study the spatial variation of the interference fields in emergent ultra-dense wireless communication networks.

Stochastic geometry has been used in the modeling of wireless networks to understand their performance characteristics (see [4]). Scaling limits of point interference has been studied as a characteristic of network performance under various modeling assumptions (see [3, 76]). Most of this prior work focuses on studying the SINR at a typical location in space. In [53], the authors have studied appropriate scaling limits of the SINR field, and have characterized the limit as a Gaussian field, under bounded path loss functions. In their paper, they use the rich set of tools and results available for Gaussian fields to study the properties of dense networks.

In this chapter, we obtain large deviation principle for the sample average of the shot-noise field, instead of considering the Gaussian scaling. Such a large deviation result can be used to determine the most likely path to a rare event. In certain examples, we will be able to calculate the large deviation rate function, and derive some implications of the LDP.

The shot-noise field also shows up in the context of queuing theory, and insurance and risk modeling. In these modeling problems, there is a notion of time, and the shot-noise response function have a causal structure. Large deviations for such models have been studied in [32, 10]. This analysis does not apply to the model studied in this chapter, where a causal structure is absent.

The main technique used to obtain the LDP is Cramer's theorem for general topological vector spaces and it is applied to the space of Radon measures. Then we use the contraction principle to transform this LDP into an LDP for a shot-noise field.

5.2 Preliminaries

In this chapter, we use slightly different notation, since we deal with signed measures. Let X be a compact subset of $\mathbb{R}^d$, for some $d > 0$. Let $C(X)$ denote the space of real valued continuous functions, equipped with the uniform norm topology. The space of Radon measures on X is the space of signed measures with finite total variation. This space is denoted by $M(X)$ and it is the dual of $C(X)$. We may equip $M(X)$ with the topology of vague convergence, which is the topology of pointwise convergence over $C(X)$. With this topology, $M(X)$ is a topological vector space, and its dual space, as a set, is $C(X)$ itself. We also note that $M(X)$ is a Polish space.

One particular metric on $M(X)$ is given by the bounded Lipschitz norm:

$$
\begin{aligned}
&d(\mu, \nu) \\
&= \|\mu - \nu\|_{BL} \\
&= \sup \left\{ \int_X f d\mu - \int_X f d\nu : |f|_\infty \leq 1, |f(x) - f(y)| \leq d(x,y) \text{ for all } x, y \in X \right\}
\end{aligned}
$$

We denote by $M_+(X)$, the cone of non-negative measures in $M(X)$.

5.3 LDP for Poisson Point Processes

We will now discuss the LDP for the densification of the Poisson point process. With an abuse of notation, let ℓ denote the Lebesgue measure on $\mathbb{R}^d$, for any dimension d. Let $\Phi_{\varepsilon^{-1}}$ denote the Poisson point process on X with intensity ε^{-1}. We would like that all the Poisson measures to be defined on a common probability space. For this, let Φ denote a Poisson point process on $X \times \mathbb{R}^+$ and let, for any measurable subset $A \subset X$

$$
\Phi_{\varepsilon^{-1}}(A) = \int_{A \times \mathbb{R}^+} \mathbb{1}_{[0, \varepsilon^{-1})}(r) \Phi(dx, dr).
$$

We are interested in an LDP for the measures $N_\varepsilon = \varepsilon \Phi_{\varepsilon^{-1}}$, for $\varepsilon > 0$. Let $I : M(X) \to [0, \infty)$ be defined as

$$
I(\mu) = \begin{cases} \int_X g \log g - g + 1 \mathrm{d}x & \text{if } g := \frac{d\mu}{d\ell} \text{ exists,} \\ \infty & \text{otherwise.} \end{cases} \tag{5.1}
$$

So, $I(\mu) = \infty$ if $\mu \notin M_+(X)$. The main result of this section is stated in the following theorem.

Theorem 5.3.1. *Let $I : M(X) \to \infty$ be as in* (5.1). *N_ε satisfies a large deviations principle in $M(X)$ with good rate function I. In particular, for any measurable set $G \subset M(X)$, we have*

$$-\inf_{\mu \in G^o} I(\mu) \leq \liminf_{\varepsilon \to 0} \varepsilon \log \mathrm{P}(N_\varepsilon \in G)$$
$$\leq \limsup_{\varepsilon \to 0} \varepsilon \log \mathrm{P}(N_\varepsilon \in G) \leq -\inf_{\mu \in \overline{G}} I(\mu).$$

Proof. We use Cramer's theorem for general topological vector spaces as presented in Theorem 6.1.3 of [18]. The dual space of $M(X)$ is space of continuous functions on X, $C(X)$. Thus, we define the log-moment generating function, Λ, as

$$\Lambda(f) := \log \mathrm{E} \exp(\int_X f(x) N_1(dx), \tag{5.2}$$

for any $f \in C(X)$. Using the formula for the Laplace transform of a Poisson point process [17], we have $\Lambda(f) := \int_X \exp(f(x)) - 1 \mathrm{d}x$.

For any $\mu \in M_+(X)$, let

$$\Lambda^*(\mu) = \sup_{f \in C(X)} \int f \mathrm{d}\mu - \Lambda(f).$$

Using Theorem 6.1.3 of [18], we have the full LDP, with rate function Λ^*, once we have proved the following statements:

1. Assumption 6.1.2(b) of [18]: For every compact $K \subset M(X)$, the closed convex hull is also compact.

2. The random variables N_ε are exponentially tight.

The first assertion can be inferred from the equivalence of vaguely relatively compact sets and vaguely bounded sets.

We now prove exponential tightness of N_ε. For $\alpha \in \mathbb{R}^+$, define

$$K_\alpha = \cap_{k=1}^{\infty} \{\mu \in M_+(X) : \mu(X) \leq \alpha \ell(X)\}.$$

K_α is a compact subset of $M(X)$, since it is closed and vaguely bounded. We have

$$\begin{aligned} \mathrm{P}(N_\varepsilon \notin K_\alpha) &\leq \frac{(\varepsilon^{-1}\ell(X))^{\lceil \alpha\varepsilon^{-1}\ell(X)\rceil}}{\lceil \alpha\varepsilon^{-1}\ell(X)\rceil!} \\ &\leq \frac{e}{\sqrt{2\pi}} \exp(-\alpha\varepsilon^{-1}\ell(X)(\log\alpha - 1)), \end{aligned} \tag{5.3}$$

where we have used the formula for the probability mass function for the Poisson distribution and the Stirling's approximation. Since the upper bound in the above expression decreases to zero as $\alpha \to \infty$, this completes the proof of exponential tightness of measures corresponding to N_ε.

We now show that $\Lambda^* = I$, where I is as defined in (5.1).

Claim 1. $\Lambda^* = I$.

Proof of Claim 1. The standard argument to show this is to use the duality theorem (Theorem 4.5.8 of [18]). Accordingly, we show first that I is convex and lower semicontinuous. It is easily seen that I is convex since the function $\tau(x) = x\log x - x + 1$ is convex. Now, to show that I is lower semicontinuous, consider the set $K'_\alpha := \{\mu \in M(X) : I(\mu) \leq \alpha\}$, for $\alpha > 0$. Let $\{\mu_n\} \subset K'_\alpha$ be sequence converging to $\mu \in M_+(X)$. Then, the functions $f_n = \frac{d\mu_n}{d\ell}$ exists and $\int_X \tau(f_n)\mathrm{d}x \leq \alpha$. Since $\tau(x)$

grows super-linearly in x, there is a $c > 0$, such that $x \leq c(\tau(x) + 1)$

$$\begin{aligned} \mu(X) &\leq \sup_n \mu_n(X) \\ &= \sup_n \int_X f_n(x) dx \\ &\leq \sup_n \int_X c(\tau(f_n) + 1) dx \leq c(\alpha + \ell(X)). \end{aligned} \tag{5.4}$$

So $\mu \in M(X)$. We now need to prove that μ is also absolutely continuous with respect to the Lebesgue measure. If $\mu(X) = 0$, we have nothing to prove. Otherwise, we use the lower semicontinuity of relative entropy as follows. Since $\mu_n \to \mu$ and $\mu(X) \neq 0$, we may assume WLOG that $\mu_n(X) > 0$, for all $n \geq 0$. Let $\overline{\mu_n} = \mu_n/\mu_n(X)$ and $\overline{\mu} = \mu/\mu(X)$. If $R(\mathrm{P}\,||\,\mathrm{Q})$ denotes the relative entropy between measures P and Q, we have

$$\begin{aligned} R(\overline{\mu_n}||\overline{\ell}) &= \frac{1}{\mu_n(X)} \int_X \left(\log(f_n(x)) + \log \frac{\ell(X)}{\mu_n(X)} \right) f_n(x)\ dx \\ &= \frac{1}{\mu_n(X)} \int_X \tau(f_n(x)) + f_n(x) - 1 + f_n(x) \log \frac{\ell(X)}{\mu_n(X)}\ dx \\ &\leq \frac{\alpha}{\mu_n(X)} + 1 - \frac{\ell(X)}{\mu_n(X)} + \log \frac{\ell(X)}{\mu_n(X)}. \end{aligned} \tag{5.5}$$

Thus, by lower-semicontinuity,

$$\begin{aligned} R(\overline{\mu}||\overline{\ell}) &\leq \liminf_{n\to\infty} R(\overline{\mu}||\overline{\ell}) \\ &\leq \frac{\alpha}{\mu(X)} + 1 - \frac{\ell(X)}{\mu(X)} + \log \frac{\ell(X)}{\mu(X)} \\ &< \infty. \end{aligned}$$

Therefore, $\mu << \ell$.

Suppose $f = \frac{d\mu}{d\ell}$. Let $h_n = (f \vee 1/n) \wedge n$, with $n \in \mathbb{N}$. Fix $n \in \mathbb{N}$. By Lusin's theorem (Theorem 7.10 [30]), for every $\epsilon > 0$ there exists $\psi_{\epsilon,n} = \log h_n$, except on a set of ℓ-measure ϵ, and $\psi_{\epsilon,n} \leq h_n$. We have, by vague convergence of $\mu_k \to \mu$,

$$\begin{aligned}
\int_X f\psi_{\epsilon,n} &= \lim_{k\to\infty} \int_X f_k \psi_{\epsilon,n}\; dx \\
&= \lim_{k\to\infty} \int_X f_k \ln f_k - f_k + 1\mathrm{d}x - \int_X e^{\psi_{\epsilon,n}} \left(\frac{f_k}{e^{\psi_{\epsilon,n}}} \ln \frac{f_k}{e^{\psi_{\epsilon,n}}} - \frac{f_k}{e^{\psi_{\epsilon,n}}} + 1 \right)\; dx \\
&\qquad\qquad + \int_X e^{\psi_{\epsilon,n}} - 1\mathrm{d}x \\
&\leq \alpha + 0 + \int_X e^{\psi_{\epsilon,n}} - 1\mathrm{d}x.
\end{aligned}$$

Thus,

$$\int_X f\psi_{\epsilon,n} - e^{\psi_{\epsilon,n}} + 1\mathrm{d}x \leq \alpha.$$

Taking $\epsilon \to 0$, we obtain that $\int_X f \log h_n - h_n + 1\mathrm{d}x \leq \alpha$. Finally, by Fatou's lemma, taking $n \to \infty$, we obtain that $\int_X f \log f - f + 1\mathrm{d}x \leq \alpha$. This implies that I is lower semicontinuous.

We now proceed to show that main property of the duality theorem, i.e., to show that for any $\psi \in C(X)$, $\Lambda(\psi) = \sup_{\mu \in M_+(X)} \int \psi(x)\mu(dx) - I(\mu)$, or equivalently,

$$\int_X e^{\psi} - 1\mathrm{d}x = \sup_{f \in L_+(X)} \int_X \psi(x) f(x)\; dx - \int_X f(x)\log f(x) - f(x) + 1\mathrm{d}x,$$

where $L_+(X)$ is the space of non-negative measurable functions on X.

Fix $\psi \in C(X)$. Taking $f = \exp\psi$, we have

$$\begin{aligned}
\int_X e^{\psi} - 1\mathrm{d}x &= \int_X \psi(x)f(x) - (f(x)\log f(x) - f(x) + 1)\mathrm{d}x \\
&\leq \sup_{f\in L_+} \int_X \psi(x) f(x)\mathrm{d}x - \int_X f(x)\log f(x) - f(x) + 1\mathrm{d}x.
\end{aligned}$$

To prove the converse inequality, we simply note that the argument in the integral in the RHS is maximized pointwise at $f(x) = e^{\psi}$ for all $x \in X$. This completes the proof the claim. $\square$

With the above claim, we may apply Theorem 6.1.3 of [18], to obtain the LDP. $\square$

5.4 LDP for Shot-Noise Process

In this section, we give a large deviation principle for average shot-noise field generated by Poisson point processes of increasing intensity. Given a response function $h : \mathbb{R}^d \to \mathbb{R}$, the shot-noise field of a finite collection of points $\Psi \subset X$ is the function

$$x \mapsto \sum_{y \in \Psi} h(x - y).$$

We are interested in the average shot-noise field

$$f(x) = \varepsilon \sum_{y \in \Phi_{\varepsilon^{-1}}} h(x - y) = \int_X h(x - y) N_\varepsilon(\mathrm{d}y).$$

Since in applications, the shot-noise response function is continuous, henceforth we assume that h is continuous.

We have the following results as a direct application of the contraction principle (Theorem 4.2.1 of [18]).

Theorem 5.4.1. *The class of functions $\{f_\varepsilon\}_\varepsilon$ satisfy an LDP, with the good rate function*

$$I'(g) := \inf\left\{\int_X \tau(\varphi)\,dx \,\middle|\, \varphi \in L^1(X), \int_X h(x-y)\varphi(y)\,dy = g(x), \forall x \in X\right\}, \quad (5.6)$$

with the understanding that the infimum over an empty set is taken as ∞.

Proof. Consider the map $M(X) \to C(X)$, given by

$$\mu \mapsto \int_X h(x-y)\mu(\mathrm{d}y).$$

This map is continuous with respect to the weak topology on $M(X)$. Indeed, for any sequence μ_n converging weakly to $\mu \in M(X)$, we have that pointwise for all $x \in X$, $\lim_{n\to\infty} \int_X h(x-y)\mu_n(\mathrm{d}y) = \int_X h(x-y)\mu(\mathrm{d}y)$. Since, h is continuous on a compact domain $X \oplus -X$, it is uniformly continuous. This necessitates that $\int_X h(x-y)\mu_n(\mathrm{d}y)$, $n \in \mathbb{N}$, is a continuous function, Since these functions convergence convergence pointwise to the continuous function $\int_X h(x-y)\mu(\mathrm{d}y)$ on a compact domain, we must have uniform convergence in the above limit.

Thus, we may apply the contraction principle, to obtain that the functions f_ε satisfy a large deviations principle with the good rate function

$$I'(g) = \inf_{\mu} \left\{ I(\mu) \middle| \int_X h(x-y)\mu(\mathrm{d}y) = g(x), \forall x \in X \right\}.$$

Since $I(\mu) < \infty$ if and only if $\mu << \ell$, we may write

$$I'(g) = \inf \left\{ \int_X \tau(\varphi)\mathrm{d}x \middle| \varphi \in L^1(X), \int_X h(x-y)\varphi(y)\mathrm{d}y = g(x), \forall x \in X \right\}.$$

This completes the proof. $\square$

5.5 Applications to Ultra-Dense Wireless Networks

Consider a cellular network where the base stations are distributed according to a homogeneous Poisson point process in $X \subset \mathbb{R}^2$, Φ_λ, with intensity λ. Let

$h : \mathbb{R}^2 \to \mathbb{R}_+$ be a continuous non-negative function. Suppose all the base stations transmit signal at power level p, and assume that the channel response function is given by h. Then, the total power received at a location $x \in X$ from all the base stations is $\mathcal{I}_\lambda(x) = p \sum_{x \in \Phi_\lambda} h(x - y)$.

Also, for any point $x \in X$, let $c_\lambda(x)$ denote the closest point in Φ_λ. Assuming that a user associates with the closest base station, the total interference seen by the user is given by

$$\mathcal{J}_\lambda(x) = p \sum_{y \in \Phi_\lambda \setminus c_\lambda(x)} h(x - y). \tag{5.7}$$

Thus, the signal-to-interference ratio (SIR) field is given by

$$\mathrm{SIR}_\lambda(x) = \frac{ph(x - c_\lambda(x))}{\mathcal{J}_\lambda(x)}, \tag{5.8}$$

and the Shannon rate field is given by

$$\mathcal{S}_\lambda(x) = w \log(1 + \mathrm{SIR}_\lambda(x)), \tag{5.9}$$

where w is the bandwidth of the transmission channel. When the intensity λ is high, we have that the signal received at a point x converges to $ph(0)$ and $\log(1+\mathrm{SIR}_\lambda(x)) \approx \mathrm{SIR}_\lambda(x)$. Thus, for large λ, $\mathcal{S}_\lambda(x) \approx \frac{wph(0)}{\mathcal{J}_\lambda(x)} \approx \frac{wph(0)}{\mathcal{I}_\lambda(x)}$. As λ increases to infinity, the interference term goes to infinity too. To balance this effect, and to obtain a constant Shannon rate, the bandwidth must increase linearly with λ. Hence, we assume that $w_\lambda = \lambda \bar{w}$. In this case,

$$\mathcal{S}_\lambda(x) \approx \frac{\lambda p \bar{w} h(0)}{\mathcal{I}_\lambda(x)}.$$

Since $\mathcal{I}_\lambda/\lambda$ is concentrated near the function $\int_X h(x-y)dy$, the Shannon rate field is concentrated around

$$\bar{\mathcal{S}}(x) := \frac{p\bar{w}h(0)}{\int_X h(x-y)dy}.$$

The large deviation result in the previous section can be used to obtain asymptotic rate at which rare events occur. For example, consider the event that $\inf_x \mathcal{S}_\lambda(x) - \bar{\mathcal{S}}(x) \geq -\epsilon$, i.e., the event that the Shannon rate drops below the expected Shannon rate by at least an ϵ somewhere in the domain. Using the LDP result in the previous section, we have that

$$\begin{aligned}
&\limsup_{\lambda\to\infty} \frac{1}{\lambda} \log \mathrm{P}(\inf_x \mathcal{S}_\lambda(x) - \bar{\mathcal{S}}(x) \leq -\epsilon) \\
&\leq -\inf\left\{ I(\mu^g) \middle| \sup_x \left(f(x) - \frac{H(x)}{1 - \frac{\epsilon}{p\bar{w}h(0)} H(x)} \right) \geq 0, \int_X h(x-y)g(x) = f(x) \right\},
\end{aligned}$$

where $H(x) = \int_X h(x-y)dy$.

5.6 Integral Equations of the First Kind

In this section, we recall a number of results from the theory of Integral Equations that will be useful in simplifying the form of the LDP in (5.6).

Consider the operator $\mathcal{H} : L^2(X) \to L^2(X)$,

$$\mathcal{H}(\phi) = \int_X h(x-y)\phi(y)dy. \tag{5.10}$$

This is a compact linear L^2 operator (see Section 2.5 of [47]). In (5.6), given a function $f \in C(X)$, we are interested in solving the so-called integral equation the first kind,

$\mathcal{H}(\phi) = f$. Since the operator $\mathcal{H}$ is compact, the problem of solving this equation is an ill-posed inverse problem in the Hadamard-sense. We now recall the Picard's theorem on solutions of the integral equation.

Let $\mathcal{H}^*$ be the adjoint of $\mathcal{H}$,

$$\mathcal{H}^* f(y) = \int_X h(x-y) f(x) dx,$$

which is also a compact linear operator. The operator $\mathcal{H}^*\mathcal{H}$ is then a non-negative compact self-adjoint linear operator. By spectral theorem for self-adjoint compact operatoes, there exists at most a countable set of positive eigenvalues accumulating at zero. Assume that the sequence $\{\lambda_n^2\}$, $\lambda_n \geq 0$, of eigenvalues are ordered such that $\lambda_i \geq \lambda_{i+1}$ and are repeated according to their multiplicities. Then, there exists orthonormal sequence of functions $\{\varphi_n\}$ and $\{g_n\}$, in $L^2(X)$ such that

$$\mathcal{H}\varphi_n = \lambda_n g_n, \quad \mathcal{H}^* g_n = \lambda_n \varphi_n \; \forall n \in \mathbb{N}.$$

For each $\varphi \in L^2(X)$, we have the *singular value decomposition*

$$\varphi = \sum_{n=1}^{\infty} \langle \varphi, \varphi_n \rangle \varphi_n + \mathcal{Q}\varphi$$

where $\mathcal{Q} : L^2(X) \to L^2(X)$ is the orthogonal projection operator onto the null-space of $\mathcal{H}$, and

$$\mathcal{H}\varphi = \sum_{n=1}^{\infty} \lambda_n \langle \varphi, \varphi_n \rangle g_n.$$

Similarly, $\{\varphi_n\}$ form one sequence of orthonormal eigenvectors of $\mathcal{H}^*\mathcal{H}$, and $g_n = \mathcal{H}\varphi_n$. Every such system $(\lambda_n, \varphi_n, g_n)$, with these properties is called a *singular system* of $\mathcal{H}$.

Moreover, the equation of the first kind $\mathcal{H}\varphi = f$ is solvable if and only if f belong to the orthogonal complement on the null-space of $\mathcal{H}^*$ and we have:

$$\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \langle f, g_n \rangle^2 < \infty.$$

In this case, all solutions are of the form

$$\varphi \in \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \langle f, g_n \rangle \varphi_n + N(\mathcal{H}),$$

where $N(\mathcal{H})$ is the null-space of $\mathcal{H}$.

We now consider an instance shot-noise field where it is easy to guess the eigenfunctions.

Example 1. Let $d = 1$, and let $X = [0, 1]$. Let

$$h(x - y) = a_0 + \sum_{n=1}^{\infty} a_n \sin(2n\pi(x - y)), \tag{5.11}$$

where $\sum_{n=1}^{\infty} a_n^2 < \infty$ and $a_i \neq 0$ for all $i \geq 0$. In this case, the functions

$$\{\sqrt{2} \sin 2n\pi x, \sqrt{2} \cos 2n\pi x\}_{n=1}^{\infty}$$

and the constant function 1 form a complete orthonormal basis of $L^2[0, 1]$. The kernel $\mathcal{H}$ maps the basis elements as follows:

$$\mathcal{H}(1) = a_0,\ \mathcal{H}(\sin 2n\pi x) = -a_n \cos(2n\pi x) \text{ and } \mathcal{H}(\cos 2n\pi x) = a_n \sin(2n\pi x).$$

Therefore, the solution of $\mathcal{H}(\varphi) = f$ exists if and only if

$$\sum_{n=1}^{\infty} a_n^{-2} (\langle g, \sin 2n\pi x \rangle^2 + \langle g, \cos 2n\pi x \rangle^2) < \infty,$$

and the solution is given by

$$\varphi(x) = a_0^{-1}\langle f, 1\rangle + \sum_{n=1}^{\infty} 2a_n^{-1}(\langle f, \sin 2n\pi(\cdot)\rangle \cos 2n\pi x - \langle g, \cos 2n\pi(\cdot)\rangle \sin 2n\pi x). \tag{5.12}$$

If the response function h is given by (5.11), the Large Deviation rate function of the average shot-noise field $I'(f) = \int_0^1 \varphi(x)(\log\varphi(x) - 1)dx + 1$ if

$$\sum_{n=1}^{\infty} a_n^{-2}(\langle f, \sin 2n\pi x\rangle^2 + \langle f, \cos 2n\pi x\rangle^2) < \infty,$$

where φ is as given in (5.12). $I'(f) = \infty$ otherwise.

In particular, for $h(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{2n\pi}\sin(2n\pi(x))$, we have the unique solution of $\mathcal{H}\varphi = f$ exists if and only if

$$n^2\langle f, \sin 2n\pi x\rangle^2 < \infty,$$

and in this case, f is differentiable almost everywhere and the solution is

$$\begin{aligned}\varphi(x) &= \langle f, 1\rangle + \sum_{n=1}^{\infty} 4n\pi(\langle f, \sin 2n\pi(\cdot)\rangle \cos 2n\pi x - \langle f, \cos 2n\pi(\cdot)\rangle \sin 2n\pi x) \\ &= 2f'(x) + \langle f, 1\rangle.\end{aligned}$$

Thus, the Large Deviation rate function is

$$I'(f) = \begin{cases} \int_0^1 (\langle f, 1\rangle + 2f'(x))[\log(\langle f, 1\rangle + 2f'(x)) - 1]dx + 1. & \text{if } f \text{ is differentiable,} \\ \infty & \text{otherwise.}\end{cases}$$

5.7 Concluding Remarks

In this chapter, we provided a large deviations principle for dense Poisson point processes, and also for the shot-noise it generates. A Large Deviation Principle allows one to characterize the most likely way in which a rare event can occur. As such, the study of the problem studied in this chapter is an essential in predictive modeling of systems determined by random point processes, or in estimation algorithms. In our future work, we would like to obtain explicit Large deviation rate function for the shot-noise field for real-world examples, and study their applications. We would also like to develop Large deviation results for other distributions and densification schemes for point processes.

Appendices

Appendix A

Appendix to Chapter 4

A.1 Dynamic reversibility of Markov processes

In this section we give a brief discussion of a result needed to construct the product form distribution of the model studied on Chapter 4 in the compact domain case. This concept will be termed dynamic reversibility of a Markov process, following the terminology in [45], where the concept was discussed for Markov processes on countable state spaces. We thus define it on countable state spaces first, and then on general state spaces.

Let $\{X(t)\}_{t\in\mathbb{R}}$ be a stationary, irreducible continuous-time Markov process with values in a countable state space S. Let $q(j,k)$ denote the transition rate from state $j \in S$ to $k \in S$ and let π be the stationary distribution of the process. In this case, the balance equations are $\sum_{j\in S} \pi(j)q(j,k) = 0$.

The reversed process, $X(-t)$, is also a stationary Markov process with transition rates $q'(j,k) = \frac{\pi(k)q(k,j)}{\pi(j)}$. The converse of this statement can be used as a characterization of the stationary distribution. We state this result in the following theorem.

Theorem A.1.1. *Let $X(t)$ be a stationary irreducible Markov process with transition rates $q(j,k)$, $j,k \in S$. If there exists a collection of numbers $q'(j,k)$, $j,k \in S$, and a* *probability measure π on S such that*

$$\pi(j)q(j,k) = \pi(k)q'(k,j), \quad j,k \in S, \tag{A.1}$$

then π is the stationary distribution of the process and q' is the transition rate matrix for the reversed process.

Thus, if we can guess the transition rates of the reversed process and a stationary measure, we can verify them by checking a local balance condition of the form eq. A.1. See Theorem 1.13 of [45] for a proof of this result. In practice, finding q' is usually as intractable as finding the stationary distribution directly. However, occasionally we may come across pairs of Markov processes that are reversed versions of each other, perhaps after a transformation of the state space. We state this phenomenon in the next theorem.

Theorem A.1.2. *Let S, T be two countable spaces. Let $X(t)$ and $Y(t)$ be two stationary irreducible Markov processes with values in S and T, and transition matrices q and q' respectively. Suppose there is an isomorphism $\phi : S \to T$ between the two spaces. Also suppose that there is a probability measure π on S such that*

$$\pi(j)q(j,k) = \pi(k)q'(\phi(k), \phi(j)), \quad j,k \in S,$$

then π is the stationary distribution of $X(t)$ and $\pi(\phi^{-1}(\cdot))$ is the stationary distribution of $Y(t)$.

Theorem A.1.2 can be stated in a more general setting, which we now state and prove.

Theorem A.1.3. *Suppose S and T be two locally compact Hausdorff topological spaces. Let $X(t)$ and $Y(t)$ be two stationary Markov jump processes with values in S and T. Suppose that probability semi-group of the process $X(t)$ ($Y(t)$) is characterized by the generators L_X (L_Y), that is defined over $\mathrm{dom}(L_X)$ ($\mathrm{dom}(L_Y)$), where the domain is a subset of the Banach space of continuous functions over S (T) vanishing at infinity, equipped with the uniform norm topology. Let $\phi : S \to T$ be a measure space isomorphism such that for all $f \in \mathrm{dom}(L_X)$, we have $f \circ \phi^{-1} \in \mathrm{dom}(L_Y)$. If π is a probability measure on S such that*

$$\int_S f(x) L_X g(x) \pi(dx) = \int_T L_Y(f \circ \phi^{-1})(y) g \circ \phi^{-1}(y) \phi_* \pi(dy),$$

then π is a stationary distribution for $X(t)$.

Proof. Let g be any element in $\mathrm{dom}(L_x)$. Taking a sequence $f_n \in \mathrm{dom}(L_Y)$ such that f_n converges pointwise to the constant function 1, as $n \to \infty$, we have

$$\int_S L_X g(x) \pi(dx) = \int_T L_Y(1)(y) g \circ \phi^{-1}(y) \phi_* \pi(dy) = 0.$$

By standard results from the theory of positive operator semi-groups it is known that for $g \in \mathrm{dom}(L_X)$ implies that the map $x \mapsto \mathrm{E}[g(X(t))|X(0) = x]$ belongs to $\mathrm{dom}(L_X)$ (see Lemma 1.3 of [24] for example). Thus, for $g \in \mathrm{dom}(L_X)$, we have

$$\frac{d}{dt} \int_S \mathrm{E}[g(X(t))|X(0) = x] \pi(dx) = \int_S L(\mathrm{E}[g(X_t)|\cdot])(x) \pi(dx) = 0.$$

It is also known that $\mathrm{dom}(L_X)$ is dense on the space of continuous functions vanishing at infinity (Theorem 1.4 of [24]). This implies that $\mathrm{E}_\pi\, g(X(t)) = \mathrm{E}_\pi\, g(X(0))$ for all bounded continuous functions and all $t > 0$, and so, π must be a stationary measure of $X(t)$. $\square$

If two processes satisfy the hypothesis of the above theorem, we say that the processes are dynamically reversible.

A.2 Some Additional Global Notation

In the following few sections we need some useful universal notation to define the transitions in the Markov processes. We collect them here in this section.

Let $\gamma = (x_1, \ldots, x_n)$, $n \in \mathbb{N}$ and x respectively be a list of elements and a particular element belonging to the same abstract space S. We define the following operators:

1. Let $\gamma \blacktriangleleft_i x$, $i = 0, \ldots, n$, denote the insertion of element x after the i-th element in γ, i.e.,
$$\gamma \blacktriangleleft_i x = (x_1, \ldots, x_i, x, x_{i+1}, \ldots, x_n).$$
2. Let $\gamma \blacktriangleright_i$, $i = 1, \ldots, n$, denote the removal of the i-th element of γ, i.e.,
$$\gamma \blacktriangleright_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).$$
3. Let $\gamma \blacktriangle_i x$ denote the replacement of the i-th element in γ with x, i.e.,
$$\gamma \blacktriangle_i x = (x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n).$$
4. In the above notation, we may drop the subscript i if $i = |\gamma|$, i.e., when we are making changes to the last element.

A.3 Continuous-Time FCFS Bipartite Matching Model with Reneging

In this section, we illustrate how dynamic reversibility is used in the proof of Theorem 4.2.2, by working on a countable state space Markov model. This allows us to organize and present the main ideas without the complexity of dealing with measure valued processes.

Specifically, in this section, we consider the following modified version of the First-come-first-serve bipartite matching model considered in [1]. Consider two finite sets of types $\mathcal{C} = \{c_1, \dots, c_I\}$ and $\mathcal{S} = \{s_1, \dots, s_J\}$ and a bipartite compatibility graph $G = (\mathcal{C}, \mathcal{S}, \mathcal{E})$ with $\mathcal{E} \subset \mathcal{C} \times \mathcal{S}$. Let $\bar{\lambda}$ be a measure on $C \cup S$, and $\mu > 0$ be a parameter. We say that c and s can be *matched* together or are *compatible* if $(c, s) \in \mathcal{E}$ in the compatibility graph E. We define the first-come-first-serve bipartite matching model with reneging as a Markov jump process with state space, Γ, which is the set of all finite ordered lists of elements from $C \cup S$ such that for every $c \in C$ and $s \in S$ in the list, $(c, s) \notin \mathcal{E}$. Further, given that the state of the process at any time t is $\gamma = (x_1, \dots, x_n)$, the state is updated with the following transition rates:

1. A new element $x \in C \cup S$ arrives at rate $\lambda(x)$. At the time of the arrival, if there is one or more elements in γ that is compatible to x, then the first such element, x_i, is removed, and we say that x and x_i are matched. If there is no such element, then x is added to the end of the list γ.

2. Each element in the list is removed at rate $\mu > 0$.

The comments and results of Sections 4.2.1 and 4.2.2 can be mirrored in this setting. We briefly review them here.

We can simulate the above process by using arrival from a Poisson point process Φ on $(C \cup S) \times \mathbb{R}$, with i.i.d. exponential marks in $\mathbb{R}^+$, and with intensity $\lambda \otimes \ell$. The base of the Poisson point process Φ encodes the arrivals of the agents, and the mark of a point encodes the time each agent is willing to wait (its *patience*), if they are accepted. We will use the following notation: for any point $x \in (C \cup S) \times \mathbb{R} \times \mathbb{R}^+$, c_x will denote its projection onto $C \cup S$, b_x will denote the second coordinate, and w_x will denote the third coordinate.

Standard coupling or Lyapunov based arguments can be used to show that this Markov process has a stationary regime. Moreover, a stationary version of the process can be constructed by using a coupling from the past scheme that uses an ergodic arrival process, Φ, which is now a Poisson point process on $(C \cup S) \times \mathbb{R}$, with marks as above. To construct the stationary regime, the notion of regeneration time of the system may be defined as follows. We say that $t \in \mathbb{R}$ is called a regeneration time of Φ if for all $x \in \Phi$ with $b_x < t$, we have $t - b_x > w_x$. The forward-time construction of the process starting from a regeneration time with empty initial conditions is clear. Moreover, if $t_1 < t_2$ are two regeneration times and η^1 and η^2 are such processes starting from t_1 and t_2 respectively, then for $t > t_2$, $\eta^1_t = \eta^2_t$. Thus, we can construct a bi-infinite stationary version, $\{\eta_t\}_{t \in \mathbb{R}}$, of this process as a factor of Φ, if we can find a sequence of regeneration times going to $-\infty$. Indeed, if $t_1 > t_2 > \cdots$ is such a sequence (and set $t_0 = \infty$), then the η_t for $t \in [t_i, t_{i-1})$, and for some $i \in \mathbb{N}$, is obtained by simulating the process starting from empty initial conditions from time t_i

until time t. An argument similar to the Lemma 4.2.1 can used to show the existence of such a sequence of regeneration times.

This coupling from the past scheme gives the definition of the matching function,

$$m : \Phi \to (C \cup S) \times \mathbb{R},$$

similar to the one defined in Section 4.2.2. For $x \in \Phi$, let $T \in \mathbb{R}^-$, be a regeneration time before b_x. The value of $m(x)$ can be set by simulating the process using Φ, starting from time T, with empty initial conditions. If x is matched to an agent $y \in \Phi$, then $m(x) = (c_y, b_y)$. Otherwise, if x reneges, then $m(x) = (c_x, b_w + w_x)$.

Given the function m, we can obtain the process η_t, since $\eta_t = (x \in \Phi : b_x \leq t < b_{m(x)})$, where the agents in the list are ordered according to their birth-times $b_\cdot$. Let $\mathtt{m}$ and $\mathtt{u}$ be additional marks, referring to whether an agent is matched or unmatched respectively. Consider the following detailed stochastic process $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$: for $t \in \mathbb{R}$,

1. Let $T_t = \min\{b_x : x \in \Phi, b_x \leq t < b_{m(x)}\}$.

2. Let $\Gamma_{\mathtt{u}} = \{(c_x, b_x, \mathtt{u}) : x \in \Phi, T_t \leq b_x \leq t < b_{m(x)}\}$ and $\Gamma_{\mathtt{m}} = \{(c_x, b_{m(x)}, \mathtt{m}) \in N : b_x \leq t, T_t \leq b_{m(x)} \leq t\}$.

3. Define $\hat{\eta}_t := ((c_x, s_x) : x \in \Gamma_{\mathtt{u}} \cup \Gamma_{\mathtt{m}})$, where s_y refers to the matched or unmatched status of an agent $y \in \Gamma_{\mathtt{u}} \cup \Gamma_{\mathtt{m}}$, and the list is ordered according to their second coordinates, $b_{(\cdot)}$.

Clearly, η_t can be obtained from $\hat{\eta}_t$ by removing the agents with marks $\mathtt{m}$. We call the process $\hat{\eta}_t$ the *Backward detailed process*, following the terminology in [1].

Since the backward detailed process at time t only depends on the points of Φ before time t, it is a stationary process. Moreover, it is a stationary version of some Markov process since for $t < s$, the state at time s can be constructed using the state at time t and the process Φ in the interval $(t, s]$. We describe this Markov process in detail now. A valid state of this Markov process can be a finite list of elements, $(x_1, \ldots, x_n)$ from the set $(C \cup S) \times \{\mathtt{u}, \mathtt{m}\}$ such that

1. $s_{x_1} = \mathtt{u}$, if $n \geq 1$.
2. If $s_{x_i} = s_{x_j} = \mathtt{u}$ then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.
3. For all $i < j$, if $s_i = \mathtt{u}$ and $s_j = \mathtt{m}$ then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

Below, we utilize the definitions in Section A.2. Additionally, mirroring our notation in the continuum setting, we define for any $x \in C \cup S$, let $N(\{x\}) = \{y \in C \cup S : (x, y) \in \mathcal{E} \text{ or } (y, x) \in \mathcal{E}\}$ and for any $A \subseteq C \cup S$ let $N(A) = \cup_{x \in A} N(\{x\})$. With an abuse of notation, we will let $N(x) := N(\{x\})$ for $x \in C \cup S$. Also, for $\gamma \in O(C \cup S, \{\mathtt{u}, \mathtt{m}\})$ and any $x \in \gamma$, we will denote

- $\gamma^x = \{y \in \gamma : y <_\gamma x\}$,
- $\gamma^{\mathtt{u}} = \{y \in \gamma : s_y = \mathtt{u}\}$ and $\gamma^{\mathtt{m}} = \{y \in \gamma : s_y = \mathtt{m}\}$,
- $$W_x = \begin{cases} N(c_x) \backslash N(\gamma^{\mathtt{u},x}) & \text{if } s_x = \mathtt{u}, \\ \emptyset & \text{otherwise.} \end{cases}$$

The transitions and transition rates for the backward detailed process are the following: Given that the state of system is $\hat{\eta} = (x_1, \ldots, x_n)$,

1. any agent $x_i \in \hat{\eta}$, with $s_{x_i} = \mathtt{u}$, loses patience. In this case, the new state is $\hat{\eta}' = \hat{\eta} \blacktriangleright_i \blacktriangleleft (c_{x_i}, \mathtt{m})$, except possibly when $i = 1$, where we need to prune all the leading matched and exchanged elements from $\hat{\eta}'$. We still denote the new state by $\hat{\eta} \blacktriangleright_i \blacktriangleleft (c_{x_i}, \mathtt{m})$, even in this case, keeping in mind the all leading matched terms must be removed. Each such transition occurs at rate μ.

2. a new agent $x = (c_x, \mathtt{u})$, $c_x \in C \cup S$, arrives and is matched to the agent $x_i \in \hat{\eta}$, with $(c_{x_i}, c_x) \in \mathcal{E}$ and $s_{x_i} = \mathtt{u}$. In this case, the new state is (a valid pruning of) $\hat{\eta} \blacktriangle_i (c_x, \mathtt{m}) \blacktriangleleft (c_{x_i}, \mathtt{m})$. This occurs at rate $\bar{\lambda}(c_x)\mathbb{1}(c_x \in W_{x_i})$.

3. a new agent $x = (c_x, \mathtt{u})$, with $c_x \in C \cup S$ arrives and is not matched to any agent. The new state is $\hat{\eta} \blacktriangleleft x$. This occurs at rate $\bar{\lambda}(c_x)\mathbb{1}(c_x \notin N(\hat{\eta}^{\mathtt{u}}))$.

We now define the *Forward detailed process*, which is the dual of the process $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$, that we denote by $\{\check{\eta}_t\}_{t\in\mathbb{R}}$. For $t \in \mathbb{R}$, define

1. Let $Y_t = \max\{b_{m(x)} : x \in \Phi, b_x \leq t < b_{m(x)}\}$.

2. Define $\Xi_{\mathtt{m}} = \{(c_x, b_{m(x)}, \mathtt{m}) : x \in \Phi, b_x \leq t < b_{m(x)}\}$, and $\Xi_{\mathtt{u}} = \{(c_x, b_x, \mathtt{u}) : t < b_x < Y_t, t < b_{m(x)}\}$.

3. Define $\check{\eta}_t = ((c_x, s_x) : x \in \Xi_u \cup \Xi_m)$, where the elements are ordered according to the second coordinates $b_{(\cdot)}$.

The forward detailed process is also a stationary version of a Markov process. Any valid state, $(x_1, \ldots, x_n)$, of this Markov process of the system satisfies:

1. $s_{x_n} = \mathtt{m}$, when $n \geq 1$.

2. If $s_{x_i} = s_{x_j} = \mathtt{m}$, then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

3. If $i < j$, $s_{x_i} = \mathtt{u}$, $s_{x_j} = \mathtt{m}$, then $(c_{x_i}, c_{x_j}) \notin \mathcal{E}$.

The transitions and the transition rates of the Markov process are given as follows: Given that the state of the system is $\check{\eta}$, the next jump occurs at rate $\bar{\lambda}(C \cup S) + Q^0_{\mathtt{m}}(\check{\eta})\mu$, where $Q^i_{\mathtt{m}}(\check{\eta})$ is the number of matched elements in $\check{\eta}$ after i-th location. Intuitively, this is so because the total rate of new arrivals is $\bar{\lambda}(C \cup S)$ and the total death rate is $Q^0_{\mathtt{m}}\mu$, since $Q^0_{\mathtt{m}}$ is the number of unmatched agents in the forward process. For the sake of brevity, let us denote $\bar{\lambda}(C \cup S) + n\mu$ by $\rho(n)$, for all $n \in \mathbb{N}$. If $\check{\eta}$ is non-empty, at each jump, to obtain the new state, we need to process the first element, x_1, in $\check{\eta}$. This is done with the following probabilities:

1. If x_1 is matched, then it is removed from $\check{\eta}$. The new state is $\check{\eta} \blacktriangleright_1$.

2. If x_1 is unmatched, then for the next state, we sample a random variable $\tau \in \mathbb{N}$ with distribution

$$\mathrm{P}(\tau = k) = \frac{\mu}{\rho(Q^k_{\mathtt{m}} + 1)} \prod_{i=1}^{k-1} \frac{\rho(Q^i_{\mathtt{m}})}{\rho(Q^i_{\mathtt{m}} + 1)},$$

   and then sample $x_{n+1}, \ldots x_\tau$ i.i.d. random unmatched elements in $\{C \cup S\}$ with distribution $\bar{\lambda}(\cdot)/\bar{\lambda}(C \cup S)$.

(a) If there is a FCFS matching x_i, $2 \leq i \leq \tau$, then set the new state to $(x_1^{\max(n,\tau)}) \blacktriangle_i (c_{x_1}, \mathtt{m}) \blacktriangleright_1$, with the understanding that all the ending unmatched agents are discarded.

(b) If there is no FCFS matching, set the new state to $(x_1^{\max(n,\tau)}) \blacktriangleleft_\tau (c_{x_1}, \mathtt{m}) \blacktriangleright_1$.

If the state is $\check{\eta} = \emptyset$, then the next jump occurs at rate $\bar{\lambda}(C \cup S)$. When a jump occurs, a random unmatched point $x_1 \in C \cup S$ is sampled with distribution $\bar{\lambda}(\cdot)/\bar{\lambda}(C \cup S)$. The next state is decided as in step (2) above, with this new $\check{\eta} = x_1$, we ignore the details here.

We have the following theorem.

Theorem A.3.1. *The two processes $\hat{\eta}_t$ and $\check{\eta}_t$ are dynamically reversible. The concerned isomorphism ϕ is the one that takes a valid state $\hat{\eta}$, reverses the order of its elements and flips the marks* $\mathtt{u}$ *and* $\mathtt{m}$*. The stationary distribution is*

$$\pi(\hat{\eta}) = K \mathbb{1}(\hat{\eta} \textit{ is valid}) \prod_{i=1}^{n} \frac{\lambda(c_{x_i})}{\rho(Q_{\mathtt{u}}^i)}$$
$$\Pi(\emptyset) = K,$$

where $\hat{\eta} = (x_1, \ldots, x_n)$ and where K is a normalizing constant.

Outline of the proof. To prove this, we start by looking at the balance equations of the form in Theorem A.1.2. Let $\hat{\eta}$ be the state of the backward detailed process, and let $\hat{\eta}'$ be a state after a valid transition. In the following, we illustrate the local balance condition eq. A.1 for only one type of transition. Other kinds of transitions can be handled similarly. Let q and q' be the transition rates of the backward and forward detailed processes respectively. Also, let $c_j(\gamma)$ denote the type of the j-th element in γ for any $\gamma \in O(C \cup S)$.

Suppose that $\hat{\eta} = (x_1, \ldots, x_n)$, $n > 0$, and that $\hat{\eta}'$ is obtained from $\hat{\eta}$ when one of the elements at x_i, at some location $i > 1$, is matched and exchanged with a new arrival $(c_x, \mathtt{u})$. In this case, $\hat{\eta}' = \hat{\eta} \blacktriangle_i (c_x, \mathtt{m}) \blacktriangleleft (c_{x_i}, \mathtt{m})$ and

$$\frac{\pi(\hat{\eta})}{\pi(\hat{\eta}')} q(\hat{\eta}, \hat{\eta}') = \lambda(c_x) \prod_{j=1}^{|\hat{\eta}|} \frac{\lambda(c_j(\hat{\eta}))}{\rho(Q_{\mathtt{u}}^j(\hat{\eta}))} \prod_{j=1}^{|\hat{\eta}'|} \frac{\rho(Q_{\mathtt{u}}^j(\hat{\eta}'))}{\lambda(c_j(\hat{\eta}'))}. \tag{A.2}$$

The first $i-1$ elements in $\hat{\eta}'$ are $x_1, \ldots, x_{i-1}$ and $|\hat{\eta}'| = |\hat{\eta}| + 1$. Moreover, $Q_{\mathtt{u}}^j(\hat{\eta}) = Q_{\mathtt{u}}^j(\hat{\eta}') + 1$ for $i \le j \le |\hat{\eta}|$. Hence, eq. A.2 simplifies to

$$\begin{aligned} \frac{\pi(\hat{\eta})}{\pi(\hat{\eta}')} q(\hat{\eta}, \hat{\eta}') &= \prod_{j=i}^{|\hat{\eta}|} \frac{\rho(Q_{\mathtt{u}}^j(\hat{\eta}'))}{\rho(Q_{\mathtt{u}}^j(\hat{\eta}') + 1)} \times \rho(Q_{\mathtt{u}}^{|\hat{\eta}'|}(\hat{\eta})) \\ &= \mathrm{P}(\tau_{\phi(\hat{\eta}')} > |\phi(\hat{\eta}')| - i) \times \rho(Q_{\mathtt{m}}^0(\phi(\hat{\eta}'))) \\ &= q'(\phi(\hat{\eta}'), \phi(\hat{\eta})). \end{aligned}$$

We claim that local balance equations for other valid transitions can also be handled similarly. This completes the proof this theorem. $\square$

A.4 Proof of Theorem 4.2.2

In this section we present detailed calculations to show that the backward detailed process, $\{\hat{\eta}_t\}_{t\in\mathbb{R}}$, and the forward detailed process, $\{\check{\eta}_t\}_{t\in\mathbb{R}}$, defined in Section 4.2.2, are dynamically reversible as jump Markov process. In turn, we are able to prove Theorem 4.2.2.

We first define the valid states of the Markov process corresponding to the forward detailed process, and present the transitions and the transition rates, since these were skipped in the discussion in Section 4.2.2.

A valid state of the forward detailed process is given by the following rules.

Definition A.4.1. $(x_1, \dots, x_n) \in O(D \times \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$ is a *valid* state of the forward detailed process if

1. $s_{x_n} = \mathtt{m}$, if $n \geq 1$,
2. For all i, j, if $s_{x_i} = s_{x_j} = \mathtt{m}$ and $d(p_{x_i}, p_{x_j}) \leq 1$, then $c_{x_i} = c_{x_j}$,
3. For all $i < j$, if $s_{x_i} = \mathtt{u}$, $s_{x_j} = \mathtt{m}$ and $d(p_{x_i}, p_{x_j}) \leq 1$, then $c_{x_i} = c_{x_j}$.

Condition 2 in the above definition essentially states that there cannot be a compatible matched pair in a valid state. This condition is equivalent to the condition that

$$\{y \in x_1, \dots x_n : s_y = \mathtt{m}\} \cap N(\{y \in x_1, \dots x_n : s_y = \mathtt{m}\}) = \emptyset.$$

This is because, if there was a violating pair at time t, such a pair could have potentially matched to each other before time t instead of matching to their present matches. Condition 3 is required since otherwise a violating pair x_i and x_j, $i < j$, the particle x_j could potentially have matched with the particle x_i instead, which arrives before the particle x_j is matched to. This condition is equivalent to the condition that for all $1 \leq j \leq n$,

$$s_{x_j} = \mathtt{m} \implies x_j \notin N(\{y \in x_1, \dots, x_i : s_y = \mathtt{u}\}).$$

The Markov process corresponding to $\{\check{\eta}_t\}$ evolves as follows. Let $\check{\eta} = (x_1, \dots, x_n)$, $n = |\check{\eta}|$, be the state of the system at some time t. The next jump occurs at rate $\rho(Q^0_{\mathtt{m}}) = 2\lambda(D) + Q^0_{\mathtt{m}}\mu$. If $\check{\eta}$ is non-empty, the first element, x_1, in the list $\check{\eta}$ is processed at the next jump according to the following rules.

1. If x_1 is matched, then it is removed from $\check{\eta}$. The new state is $\check{\eta} \blacktriangleright_1$.

2. If x_1 is unmatched, then for the next state, we sample a random variable $\tau \in \mathbb{N}$ with distribution

$$\mathrm{P}(\tau = k) = \frac{\mu}{\rho(Q^k_{\mathtt{m}} + 1)} \prod_{i=1}^{k-1} \frac{\rho(Q^i_{\mathtt{m}})}{\rho(Q^i_{\mathtt{m}} + 1)},$$

   and then sample $x_{n+1}, \dots x_\tau$ i.i.d. random unmatched points in $D \times \boldsymbol{C}$ with distribution $\lambda \otimes m_c/(2\lambda(D))$.

   (a) If there is a first-in-first-match x_i, $2 \leq i \leq \tau$, for x_1, then set the new state to $(x_1^{\max(n,\tau)}) \blacktriangle_i (p_{x_1}, c_{x_1}, \mathtt{m}) \blacktriangleright_1$, with the understanding that all the unmatched particles at the end of the list are discarded.

   (b) If there is no FIFM matching, then set the new state to $(x_1^{\max(n,\tau)}) \blacktriangleleft_\tau (p_{x_1} c_{x_1}, \mathtt{m}) \blacktriangleright_1$.

If the state is $\check{\eta} = \emptyset$, then the next jump occurs at rate $\rho(0) = 2\lambda(D)$. A random unmatched particle x_1 is sampled from with distribution $\bar{\lambda}/2\lambda(D)$. The next state is decided as in step (2) above with $\check{\eta} = (x_1)$.

We are now ready to prove Theorem 4.2.2.

Proof of Theorem 4.2.2. To obtain the stationary distribution, we check the conditions of Theorem A.1.3. The space $O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$ is viewed as a subset of $\sqcup_{n=0}^{\infty}(D \times \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})^n$, and we use the induced topology on $O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$. With this topology, $O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$ is a locally compact Hausdorff space.

Let $\bar{D} = D \times \boldsymbol{C}$ and let $\bar{\lambda}$ be the measure $\lambda \otimes m_c$ on $\bar{D}$. Also, let $\hat{D} = D \times \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\}$ and let $\hat{\lambda} = \lambda \otimes m_c \otimes m_c$ on $\hat{D}$.

The probability semi-group of the processes $\hat{\eta}$ acts over the Banach space of continuous functions that vanish at infinity, where we use the uniform norm topology. Moreover, the generator of $\hat{\eta}$ can at least be defined on the space of compactly supported continuous functions, and has the form:

$$\begin{aligned} L_1 f(\hat{\eta}) = &\sum_{i=1}^{|\hat{\eta}|} \mathbb{1}(s_{x_i} = \mathtt{u})\mu[f(\hat{\eta} \blacktriangleright_i \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})) - f(\hat{\eta})] \\ &+ \int_{\bar{D}} \sum_{i=1}^{|\hat{\eta}|} \mathbb{1}(x \in W_{x_i})[f(\hat{\eta} \blacktriangle_i (p_x, c_x, \mathtt{m}) \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})) - f(\hat{\eta})]\bar{\lambda}(dx) \\ &+ \int_{\bar{D}} \mathbb{1}(x \notin N(\hat{\eta}^{\mathtt{u}}))[f(\hat{\eta} \blacktriangleleft (p_x, c_x, \mathtt{u})) - f(\hat{\eta})]\bar{\lambda}(dx). \end{aligned} \tag{A.3}$$

Similarly, it can also be seen that the generator of $\check{\eta}$ can also be defined over the space of compactly supported continuous functions, and the value of the generator $L_2 g(\check{\eta})$ is the sum of the following terms (in the order of the transitions listed earlier):

- (1): $\rho(Q^0_{\mathtt{m}}(\check{\eta}))\mathbb{1}(s_{x_1} = \mathtt{m})[g(\check{\eta} \blacktriangleright_1) - g(\check{\eta})]$,

- (2a):

$$\rho(Q^0_{\mathtt{m}}(\check{\eta}))\mathbb{1}(s_{x_1}=\mathtt{u})$$
$$\times\left(\sum_{k=2}^{|\check{\eta}|}\mathrm{P}(\tau(\check{\eta})>k)\mathbb{1}((p_{x_1},c_{x_1})\in W_{x_k})[g(\hat{\eta}\blacktriangle_k(p_{x_1},c_{x_1},\mathtt{m})\blacktriangleright_1)-g(\hat{\eta})]\right.$$
$$+\sum_{k=|\check{\eta}|+1}^{\infty}\mathrm{P}(\tau(\check{\eta})>k)\int_{\bar{D}^{k-|\check{\eta}|}}\big(\mathbb{1}((p_{x_1},c_{x_1})\in W_{x_k})$$
$$\left.\times[g((x_1^k)\blacktriangle_k(p_{x_1},c_{x_1},\mathtt{m})\blacktriangleright_1)-g(\hat{\eta})]\big)\,\bar{\lambda}(dx^k_{|\check{\eta}|+1})\right),$$

where we have set $s_{x_j}=\mathtt{u}$ for all $j>|\check{\eta}|$.

- (2b):

$$\rho(Q^0_{\mathtt{m}}(\check{\eta}))\mathbb{1}(s_{x_1}=\mathtt{u})$$
$$\times\left(\sum_{k=1}^{|\check{\eta}|}\mathrm{P}(\tau=k)\mathbb{1}((p_{x_1},c_{x_1})\notin N(\check{\eta}_1^{k,\mathtt{u}}))[g(\hat{\eta}\blacktriangleleft_k(p_{x_1},c_{x_1},\mathtt{m})\blacktriangleright_1)-g(\hat{\eta})]\right.$$
$$+\sum_{k=|\check{\eta}|+1}^{\infty}\mathrm{P}(\tau=k)\int_{\bar{D}^{k-|\check{\eta}|}}\mathbb{1}((p_{x_1},c_{x_1})\notin N(x_1^{k,\mathtt{u}}))$$
$$\left.\times[g((x_1^k)\blacktriangleleft(p_{x_1},c_{x_1},\mathtt{m})\blacktriangleright_1)-g(\check{\eta})]\bar{\lambda}(dx^k_{|\check{\eta}|+1})\right),$$

where $s_{x_j}=\mathtt{u}$ for all $j>|\check{\eta}|$.

When $\check{\eta}=\emptyset$, we have the following terms in $L_2g(\emptyset)$.

- (2a.):

$$\rho(0)\int_{\bar{D}}\sum_{k=2}^{\infty}\mathrm{P}(\tau(\emptyset)>k-1)\int_{\bar{D}^{k-1}}\big(\mathbb{1}((p_{x_1},c_{x_1})\in W_{x_k})$$
$$\times[g((x_2^{k-1})\blacktriangleleft(p_{x_1},c_{x_1},\mathtt{m}))-g(\emptyset)]\big)\,\bar{\lambda}(dx_2^k)\bar{\lambda}(dx_1),$$

where, $s_{x_j} = \mathtt{u}$ for all $j > 0$.

- (2b.):

$$\rho(0) \int_{\bar{D}} \sum_{k=1}^{\infty} \mathrm{P}(\tau_1 = k) \int_{\bar{D}_{\mathtt{u}}^{k-1}} \Big(\mathbb{1}((p_{x_1}, c_{x_1}) \notin N(x_1^{k,\mathtt{u}})) \\ \times [g((x_2^k) \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m})) - g(\emptyset)] \Big) \bar{\lambda}(dx_2^k) \bar{\lambda}(dx_1)$$

where, $s_{x_j} = \mathtt{u}$ for all $j > 0$.

For the product form distribution, we check the balance condition of Theorem A.1.3, with an appropriate measure space isomorphism ϕ. The isomorphism is given by the function revx, defined in Section 4.2.2, that reverses the order and flips marks $\mathtt{u}$ and $\mathtt{m}$ of a valid state. In the following, let $\phi :=$ revx denote this function.

Taking any two compactly supported continuous functions f, g, and taking each term of $\int gL_1 f + gf \mathrm{d}\hat{\pi}(\hat{\eta})$, we show that it corresponds to a few terms in $\int L_2(g \circ \phi^{-1})(\phi(\hat{\eta}))f(\hat{\eta}) + gf \mathrm{d}\hat{\pi}(\hat{\eta})$, so that the sum of these expressions is equal. In particular, the following steps suffice.

1. Take the first summation term of $\int gL_1 f + gf \ \mathrm{d}\hat{\pi}$ when it is expanded using eq. A.3. Take i-th term, with $i > 1$. Set $\hat{\eta}' = \hat{\eta} \blacktriangleright_i \blacktriangleleft (p_i, c_i, \mathtt{m})$. Let $n := |\hat{\eta}|$, and so $n = |\hat{\eta}'|$. Also, let $\hat{\eta} = (x_1, \ldots, x_n)$ and $\hat{\eta}' = (x_1', \ldots, x_n')$. The corresponding

term is

$$\begin{aligned}
&\int \mu \mathbb{1}(s_{x_i} = \mathtt{u}) g(\hat\eta) f(\hat\eta') \hat\pi(\hat\eta) \\
&:= \mu \sum_{n=i}^{\infty} \int_{\hat D^n} \mathbb{1}(s_{x_i} = \mathtt{u}) \hat\pi(\hat\eta) g(\hat\eta) f(\hat\eta') \hat\lambda(\mathrm{d}\hat\eta) \\
&= \sum_{n=i}^{\infty} \int_{\hat D^n} \hat\pi(\hat\eta') \frac{\mu \hat\pi(\hat\eta' \blacktriangleleft_{i-1} (p_{x'_n}, c_{x'_n}, \mathtt{u}) \blacktriangleright)}{\hat\pi(\hat\eta')} \\
&\qquad\qquad \times g(\hat\eta' \blacktriangleleft_{i} (p_{x'_n}, c_{x'_n}, \mathtt{u}) \blacktriangleright) f(\hat\eta') \hat\lambda(\mathrm{d}\hat\eta') \\
&= \int \mathbb{1}(n \geq i) \, \mathrm{P}(\tau_{\phi(\hat\eta')} = n - i + 1) \rho(Q^0_{\mathtt{m}}(\phi(\hat\eta'))) \\
&\qquad\qquad \times g(\hat\eta' \blacktriangleleft_{i} (p_{x'_n}, c_{x'_n}, \mathtt{u}) \blacktriangleright) f(\hat\eta') \hat\pi(\hat\eta'),
\end{aligned} \tag{A.4}$$

where in the second equality, we have used that $\hat\eta = \hat\eta' \blacktriangleleft_{i-1} (p_{x'_n}, c_{x'_n}, \mathtt{u}) \blacktriangleright$, and in the third equality, we use

$$\begin{aligned}
\frac{\mu \pi(\hat\eta' \blacktriangleleft_{i-1} (p'_n, c'_n, \mathtt{u}) \blacktriangleright)}{\pi(\hat\eta')} &= \frac{\mu}{\rho(N^i_{\mathtt{u}}(\hat\eta))} \prod_{j=i}^{n-1} \frac{\rho(N^j_{\mathtt{u}}(\hat\eta'))}{\rho(N^{j+1}_{\mathtt{u}}(\hat\eta))} \rho(N^n_{\mathtt{u}}(\hat\eta')) \\
&= \mathrm{P}(\tau_{\phi(\hat\eta')} = n - i + 1) \rho(N^0_{\mathtt{m}}(\phi(\hat\eta'))).
\end{aligned} \tag{A.5}$$

Similarly, in the first summation term of $\int gL_1 f + gf$, taking $i = 1$, and letting $k(\hat\eta)$ be the maximum element such that $x_2, \ldots, x_k$ are all matched in $\hat\eta$, we have

$$\begin{aligned}
&\int \mu g(\hat\eta) f(\hat\eta') \hat\pi(\hat\eta) \\
&= \mu \sum_{n=2}^{\infty} \int_{\hat D^n} \sum_{j=1}^{n-1} \mathbb{1}(k(\hat\eta) = j) g(x_1^n) f(x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m})) \hat\pi(\hat\eta) \hat\lambda(\mathrm{d}\hat\eta) \\
&\quad + \mu \sum_{n=1}^{\infty} \int_{\hat D^n} \mathbb{1}(k(\hat\eta) = n) g(x_1^n) f(\emptyset) \hat\pi(\hat\eta) \hat\lambda(\hat\eta),
\end{aligned}$$

where we have used the fact that if $k(\hat\eta) = |\hat\eta|$, then $\hat\eta' = \emptyset$. Consider the first term in the above expression. Setting $m = n - j + 1$ and $x'^m_1 = (x^n_{j+1} \blacktriangleleft$

$(p_{x_1}, c_{x_1}, \mathtt{m}))$, we obtain:

$$\begin{aligned}
&\mu \sum_{n=2}^{\infty} \int_{\hat{D}^n} \sum_{j=1}^{n-1} \mathbb{1}(k(\hat{\eta}) = j) g(x_1^n) f(x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m})) \hat{\pi}(\hat{\eta}) \hat{\lambda}(\mathrm{d}\hat{\eta}) \\
&= \mu \sum_{j=1}^{\infty} \sum_{m=2}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x'_m} = \mathtt{m}) \bar{\lambda}(\bar{D})^{j-1} \, \mathrm{E}_{X_2^j} \Big[g((p_{x'_m}, c_{x'_m}, \mathtt{u}) X_2^j {x'}_1^{m-1}) \\
&\qquad\qquad f({x'}_1^m) \hat{\pi}((p_{x'_m}, c_{x'_m}, \mathtt{u}) X_2^j {x'}_1^{m-1}) \Big] \hat{\lambda}(d{x'}_1^m),
\end{aligned} \tag{A.6}$$

where X_2^j are i.i.d. particles in $\tilde{D}$ with marks $\mathtt{m}$, with distribution $\bar{\lambda}(\cdot)/\bar{\lambda}(\tilde{D})$. Using a similar computation as in eq. A.5, it is easy to see that RHS of eq. A.7 is

$$\begin{aligned}
&\sum_{m=2}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x'_m} = \mathtt{m}) \rho(Q_{\mathtt{m}}^0(\phi({x'}_1^m))) \sum_{j=1}^{\infty} \mathrm{P}(\tau(\phi({x'}_1^m)) = m + j - 1) \hat{\pi}({x'}_1^m) \\
&\qquad \times \mathrm{E}_{X_2^j} \Big[g((p_{x'_m}, c_{x'_m}, \mathtt{u}) X_2^j {x'}_1^{m-1}) f({x'}_1^m) \Big] \hat{\lambda}(d{x'}_1^m).
\end{aligned} \tag{A.7}$$

Similarly, the second term in the eq. A.6 is

$$\begin{aligned}
&\mu \sum_{n=1}^{\infty} \int_{\hat{D}^n} \mathbb{1}(k(\hat{\eta}) = n) g(x_1^n) f(\emptyset) \hat{\pi}(\hat{\eta}) \hat{\lambda}(\hat{\eta}) \\
&= \hat{\pi}(\emptyset) \rho(0) \sum_{j=1}^{\infty} \mathrm{P}(\tau(\emptyset) = j) \, \mathrm{E}_{X_1^j} f(\emptyset) g(X_1^j),
\end{aligned} \tag{A.8}$$

where, X_2^j are as before, and X_1 is an independent sample from $\bar{D}$ with mark $\mathtt{u}$.

2. Now, take the second summation in $\int g L_1 f + g f \hat{\pi}$, and the i-th term, $i > 1$, in

that summation. Using a similar computation as in previous item, we obtain:

$$\begin{aligned}
&\int \int_{\bar{D}} \mathbb{1}(s_{x_i} = \mathtt{u}) \mathbb{1}\left((p,c) \in W_{x_i}\right) f(\hat{\eta}') g(\hat{\eta}) \bar{\lambda}(\mathrm{d}p, \mathrm{d}c) \mathrm{d}\hat{\pi}(\hat{\eta}) \\
&= \sum_{n=i}^{\infty} \int_{\hat{D}^n} \int_{\bar{D}} \mathbb{1}(s_{x_i} = \mathtt{u}) \mathbb{1}\left((p,c) \in W_{x_i}\right) \\
&\qquad\qquad f(x_1^n \blacktriangle_i (p, c, \mathtt{m}) \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})) g(x_1^n) \hat{\pi}(x_1^n) \bar{\lambda}(\mathrm{d}p, \mathrm{d}c) \hat{\lambda}(\mathrm{d}\hat{\eta}).
\end{aligned}$$

Setting ${x'}_1^{n+1} = x_1^n \blacktriangle_i (p, c, \mathtt{m}) \blacktriangleleft (p_{x_i}, c_{x_i}, \mathtt{m})$ and $m = n+1$, we have the RHS of the above equation is equal to

$$\begin{aligned}
&\sum_{m=i+1}^{\infty} \int_{\hat{D}^m} \mathbb{1}(s_{x'_i} = \mathtt{m} = s_{x'_m}) \mathbb{1}((p_{x'_m}, c_{x'_m}) \in W_{x'_i}) \\
&\qquad f({x'}_1^m) g({x'}_1^{m-1} \blacktriangle_i (p_{x'_m}, c_{x'_m}, \mathtt{u})) \hat{\pi}({x'}_1^{m-1} \blacktriangle_i (p_{x'_m}, c_{x'_m}, \mathtt{u})) \hat{\lambda}(d{x'}_1^m) \\
&= \sum_{m=i+1}^{\infty} \int_{\hat{D}^m} \rho(Q^0_{\mathtt{m}}(\phi({x'}_1^m)))\, \mathrm{P}(\tau(\phi({x'}_1^m) > m - i)) \mathbb{1}(s_{x'_i} = \mathtt{m} = s_{x'_m}) \\
&\qquad \mathbb{1}((p_{x'_m}, c_{x'_m}) \in W_{x'_i}) f({x'}_1^m) g({x'}_1^{m-1} \blacktriangle_i (p_{x'_m}, c_{x'_m}, \mathtt{u})) \hat{\pi}({x'}_1^m) \hat{\lambda}(d{x'}_1^m).
\end{aligned} \tag{A.9}$$

Similarly, taking first term in the second summation, and letting $k(\hat{\eta})$ as before, we have

$$\begin{aligned}
&\int \int_{\bar{D}} \mathbb{1}((p,c) \in W_{x_1}) f(\hat{\eta}') g(\hat{\eta}) \bar{\lambda}(\mathrm{d}p, \mathrm{d}c) \mathrm{d}\hat{\pi}(\hat{\eta}) \\
&= \sum_{n=2}^{\infty} \int_{\hat{D}^n} \int_{\bar{D}} \sum_{j=1}^{n-1} \mathbb{1}(k(\hat{\eta}) = j) \mathbb{1}((p,c) \in W_{x_1}) \\
&\qquad\qquad f(x_{j+1}^n \blacktriangleleft (p_{x_1}, c_{x_1}, \mathtt{m})) g(x_1^n) \hat{\pi}(x_1^n) \bar{\lambda}(\mathrm{d}p, \mathrm{d}c) \hat{\lambda}(dx_1^n) \\
&+ \sum_{n=1}^{\infty} \int_{\hat{D}^n} \int_{\bar{D}} \mathbb{1}(k(\hat{\eta}) = n) \mathbb{1}((p,c) \in W_{x_1}) f(\emptyset) g(x_1^n) \hat{\pi}(x_1^n) \tilde{\lambda}(\mathrm{d}p, \mathrm{d}c) \hat{\lambda}(dx_1^n).
\end{aligned} \tag{A.10}$$

Computations similar to the one in eqs. A.7 and A.8 show that the above is equal to

$$\begin{aligned}
&\sum_{m=2}^{\infty}\int_{\hat{D}^m}\mathbb{1}(s_{x'_m}=\mathtt{m})\rho(Q^0_{\mathtt{m}}(\phi({x'}^m_1)))\sum_{j=1}^{\infty}\mathrm{P}(\tau(\phi({x'}^m_1))>m+j-1)\\
&\qquad\times\mathrm{E}_{X_2^{j+1}}\begin{bmatrix}{\scriptstyle \mathbb{1}((p_{x_1},c_{x_1})\in W_{X_{j+1}})}\\ {\scriptstyle \times g((p_{x'_m},c_{x'_m},\mathtt{u})X_2^j{x'}^{m-1}_1)}\end{bmatrix}f({x'}^m_1)\hat{\pi}({x'}^m_1)\hat{\lambda}(d{x'}^m_1)\\
&+\hat{\pi}(\emptyset)\rho(0)\sum_{j=1}^{\infty}\mathrm{P}(\tau(\emptyset)>j)\,\mathrm{E}_{X_1^{j+1}}\,\mathbb{1}((p_{X_1},c_{X_1})\in W_{X_{j+1}})f(\emptyset)g(X_1^j),
\end{aligned}\tag{A.11}$$

where X_1 and X_{j+1} are i.i.d. with marks $\mathtt{u}$ and X_2^j are i.i.d with marks $\mathtt{m}$.

3. Finally,

$$\begin{aligned}
&\int\int_{\bar{D}}\mathbb{1}((p,c)\notin N(\hat{\eta}^{\mathtt{u}}))g(\hat{\eta})f(\hat{\eta}')\bar{\lambda}(\mathrm{d}p,\mathrm{d}c)\mathrm{d}\hat{\pi}(\hat{\eta})\\
&=\sum_{n=0}^{\infty}\int_{\hat{D}^n}\int_{\bar{D}}\mathbb{1}((p,c)\notin N(x\in x_1^n:s_x=\mathtt{u}))\\
&\qquad\qquad g(x_1^n)f(x_1^n\blacktriangleleft(p,c,\mathtt{u}))\hat{\pi}(x_1^n)\bar{\lambda}(\mathrm{d}p,\mathrm{d}c)\hat{\lambda}(dx_1^n)
\end{aligned}$$

Setting $m=n+1$, ${x'}^m_1=x_1^n\blacktriangleleft(p,c,\mathtt{u})$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\int_{\hat{D}^m}\mathbb{1}(s_{x'_m}=\mathtt{u})g({x'}^{m-1}_1)f({x'}^m_1)\hat{\pi}({x'}^{m-1}_1)\hat{\lambda}(d{x'}^m_1)\\
&=\sum_{m=1}^{\infty}\int_{\bar{D}^m}\rho(Q^0_{\mathtt{u}}(\phi({x'}^m_1)))\mathbb{1}(s_{x'_m}=\mathtt{u})g({x'}^{m-1}_1)f({x'}^m_1)\hat{\pi}({x'}^{m-1}_1)\bar{\lambda}(d{x'}^m_1).
\end{aligned}\tag{A.12}$$

Since summing over the RHS of equations A.4 to A.12, for all $i\geq 1$, we get $\int fL_2g+fg\mathrm{d}\hat{\pi}(\eta)$, we conclude that the two Markov processes $\hat{\eta}_t$ and $\check{\eta}_t$ are dynamically reversible with respect to $\hat{\pi}$. $\square$

A.5 Proof of Lemma 4.2.4

The proof is by induction on m. For $m = 1$, we need to show that

$$\sum_{\sigma \in P(n,1)} \prod_{i=1}^{n+1} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} = \frac{1}{\beta_1} \prod_{i=1}^{n} \frac{1}{\alpha_i}. \tag{A.13}$$

We use induction on n to prove eq. A.13. For $n = 1$, this is clear, since

$$\frac{1}{(\alpha_1)(\alpha_1 + \beta_1)} + \frac{1}{(\beta_1)(\beta_1 + \alpha_1)} = \frac{1}{(\alpha_1)(\beta_1)}$$

Let the length of the sequence α be n. Assuming the inductive hypothesis for eq. A.13, we have

$$\begin{aligned}
&\sum_{k=1}^{n+1} \sum_{\substack{\sigma \in P(n,1), \\ \sigma(k)-\sigma(k-1)=(0,1)}} \prod_{i=1}^{n+1} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} \\
&= \left(\sum_{k=1}^{n} \sum_{\substack{\sigma \in P(n,1), \\ \sigma(k)-\sigma(k-1)=(0,1)}} \prod_{i=1}^{n} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} + \prod_{i=1}^{n} \frac{1}{\alpha_i} \right) \frac{1}{\alpha_n + \beta_1} \\
&= \left(\sum_{\sigma \in P(n-1,1)} \prod_{i=1}^{n} \frac{1}{\alpha_{\sigma_x(i)} + \beta_{\sigma_y(i)}} + \prod_{i=1}^{n} \frac{1}{\alpha_i} \right) \frac{1}{\alpha_n + \beta_1} \\
&= \left(\prod_{i=1}^{n-1} \frac{1}{\alpha_i} \right) \frac{1}{\alpha_n + \beta_1} \left(\frac{1}{\beta_1} + \frac{1}{\alpha_n} \right) \\
&= \frac{1}{\beta_1} \left(\prod_{i=1}^{n} \frac{1}{\alpha_i} \right),
\end{aligned}$$

where in the first step we have grouped the first n terms together. This finishes the proof of the base case (eq. A.13) for the induction on m.

Now, suppose that the length of the sequence β is equal to m, $m > 1$. Let $P_k(n, m-1)$, $0 \leq k \leq n$, denote the set of paths in $P(n, m-1)$ where the first $(0, 1)$

jump is at location k. We have the following decomposition of the summation

$$\sum_{\sigma\in P(n,m)}\prod_{i=1}^{n+m}\frac{1}{\alpha_{\sigma_x(i)}+\beta_{\sigma_y(i)}}=\sum_{k=1}^{n+1}\sum_{P_k(n,m-1)}\sum_{r=0}^{k-1}\prod_{i=1}^{n+m}\frac{1}{\alpha_{\sigma_x^r(i)}+\beta_{\sigma_y^r(i)}},$$

where σ^r denotes the path obtained by inserting a $+(0,1)$ jump in the r-th location of σ. Thus,

$$\sum_{\sigma\in P(n,m)}\prod_{i=1}^{n+m}\frac{1}{\alpha_{\sigma_x(i)}+\beta_{\sigma_y(i)}}=\sum_{k=1}^{n+1}\sum_{P_k(n,m-1)}\sum_{r=0}^{k-1}\prod_{i=1}^{k}\frac{1}{\alpha_{\sigma_x^r(i)}+\beta_{\sigma_y^r(i)}}\prod_{i=k+1}^{n+m}\frac{1}{\alpha_{\sigma_x^r(i)}+\beta_{\sigma_y^r(i)}}$$

The inner-most summation in the above expression is the $(n,1)$ case of this lemma, and therefore, by the induction hypothesis, we get that the

$$\prod_{i=k+1}^{n+m}\frac{1}{\alpha_{\sigma_x^k(i)}+\beta_{\sigma_y^k(i)}}\times\frac{1}{\beta_1}\prod_{i=1}^{k-1}\frac{1}{\alpha_i}=\frac{1}{\beta_1}\prod_{i=1}^{n+m-1}\frac{1}{\alpha_{\sigma_x(i)}+\beta'_{\sigma_y(i)}},$$

where β' is the sequence of length $m-1$ with $\beta'_i=\beta_{i+1}$, for $1\le i\le m-1$. Therefore,

$$\begin{aligned}\sum_{\sigma\in P(n,m)}\prod_{i=1}^{n+m}\frac{1}{\alpha_{\sigma_x(i)}+\beta_{\sigma_y(i)}}&=\frac{1}{\beta_1}\sum_{k=0}^{n}\sum_{P_k(n,m-1)}\prod_{i=1}^{n+m-1}\frac{1}{\alpha_{\sigma_x(i)}+\beta'_{\sigma_y(i)}}\\&=\frac{1}{\beta_1}\sum_{\sigma\in P(n,m-1)}\prod_{i=1}^{n+m-1}\frac{1}{\alpha_{\sigma_x(i)}+\beta'_{\sigma_y(i)}}\\&=\prod_{i=1}^{n}\frac{1}{\alpha_i}\prod_{i=1}^{m}\frac{1}{\beta_i},\end{aligned}$$

by the induction hypothesis. This completes the proof the lemma.

A.6 Proof of Theorem 4.2.7

Let $A=\xi\backslash\gamma$, $B=\gamma\backslash\xi$ and $C=\xi\cap\gamma$. The statement of the theorem is equivalent to showing that

$$\tilde{\Pi}(A\cup B\cup C)\tilde{\Pi}(C)\ge\tilde{\Pi}(A\cup C)\tilde{\Pi}(B\cup C). \tag{A.14}$$

Let $\bar{C}$ denote another copy of the C, where we add over-lines to the particles to distinguish them from particles of C. Also, for any $E, \gamma \subseteq D \times \boldsymbol{C}$, let $N_E(\gamma) = N(\gamma \cap E)$.

Using the auxiliary Lemma 4.2.4, the LHS of the inequality above can be expressed as

$$\begin{aligned}
&\tilde{\Pi}(A \cup B \cup C)\tilde{\Pi}(\bar{C}) \\
&= \sum_{\substack{\boldsymbol{e} \in \mathcal{P}(A \cup B \cup C) \\ \boldsymbol{d} \in \mathcal{P}(\bar{C})}} \prod_{i=1}^{n+m+k} \frac{1}{\bar{\lambda}(N(\boldsymbol{e}_1^i)) + i\mu} \prod_{j=1}^{k} \frac{1}{\bar{\lambda}(N(\boldsymbol{d}_1^j)) + j\mu} \\
&= \sum_{\substack{\boldsymbol{e} \in \mathcal{P}(A \cup B \cup C) \\ \boldsymbol{d} \in \mathcal{P}(\bar{C})}} \sum_{\sigma \in P(n+m+k,k)} \prod_{i=1}^{n+m+2k} \frac{1}{\bar{\lambda}(N(\boldsymbol{e}_1^{\sigma_x(i)})) + \bar{\lambda}(N(\boldsymbol{d}_1^{\sigma_y(i)})) + i\mu},
\end{aligned} \tag{A.15}$$

where we use $\sigma_x(i),\ \sigma_y(i)$ to represent the first and second coordinates of $\sigma(i)$.

Since

$$\bar{\lambda}(N_{A \cup B \cup C}(\gamma)) \leq \begin{pmatrix} \bar{\lambda}(N_A(\gamma)) + \bar{\lambda}(N_B(\gamma)) + \bar{\lambda}(N_C(\gamma)) \\ -\bar{\lambda}(N_A(\gamma) \cap N_C(\gamma)) - \bar{\lambda}(N_B(\gamma) \cap N_C(\gamma)) \end{pmatrix}, \tag{A.16}$$

we claim that eq. A.15 is greater than

$$\sum_{\substack{\boldsymbol{e} \in \mathcal{P}(A \cup B \cup C) \\ \boldsymbol{d} \in \mathcal{P}(\bar{C}) \\ \sigma \in P(n+m+k,k)}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N_A(\boldsymbol{e}_1^{\sigma_x(i)})) + \bar{\lambda}(N_B(\boldsymbol{e}_1^{\sigma_x(i)})) + \bar{\lambda}(N_C(\boldsymbol{e}_1^{\sigma_x(i)})) + \bar{\lambda}(N_{\bar{C}}(\boldsymbol{d}_1^{\sigma_y(i)})) \\ -\bar{\lambda}(N_A(\boldsymbol{e}_1^{\sigma_x(i)}) \cap N_C(\boldsymbol{e}_1^{\sigma_x(i)})) - \bar{\lambda}(N_B(\boldsymbol{e}_1^{\sigma_x(i)}) \cap N_C(\boldsymbol{e}_1^{\sigma_x(i)})) + i\mu \end{pmatrix}^{-1}. \tag{A.17}$$

Let $P(n, m, k, k)$ be the set of all increasing vertex paths from $(0, 0, 0, 0)$ to (n, m, k, k) in $\mathbb{Z}^4$, $\sigma(0) = (0, 0, 0, 0)$ and $\sigma(n + m + 2k) = (n, k, k, k)$, for all $\sigma \in P(n, m, k, k)$. We denote the coordinates of $\sigma \in \mathbb{Z}^4$ by $(\sigma_x, \sigma_y, \sigma_z, \sigma_w)$. Using the canonical bijection between

$$\mathcal{P}(A \cup B \cup C \cup \bar{C}) \times P(n+m+k, k) \text{ and}$$
$$\mathcal{P}(A) \times \mathcal{P}(B) \times \mathcal{P}(C) \times \mathcal{P}(\bar{C}) \times P(n, m, k, k),$$

we see that eq. A.17 is equal to

$$\sum_{\substack{\boldsymbol{a}\in\mathcal{P}(A),\boldsymbol{b}\in\mathcal{P}(B)\\ \boldsymbol{c}\in\mathcal{P}(C),\bar{\boldsymbol{c}}\in\mathcal{P}(\bar{C})\\ \sigma\in P(n,m,k,k)}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)}))+\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)}))+\bar{\lambda}(N(\boldsymbol{c}_1^{\sigma_z(i)}))+\bar{\lambda}(N(\bar{\boldsymbol{c}}_1^{\sigma_w(i)})) \\ -\bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)})\cap N(\boldsymbol{c}_1^{\sigma_z(i)}))-\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)})\cap N(\boldsymbol{c}_1^{\sigma_z(i)}))+i\mu \end{pmatrix}^{-1}. \tag{A.18}$$

Applying similar reductions to the RHS of eq. A.14, we note that the result follows if we prove

$$\begin{aligned} &\sum_{\substack{\boldsymbol{a}\in\mathcal{P}(A),\boldsymbol{b}\in\mathcal{P}(B)\\ \boldsymbol{c}\in\mathcal{P}(C),\bar{\boldsymbol{c}}\in\mathcal{P}(\bar{C})\\ \sigma\in P(n,m,k,k)}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)}))+\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)}))+\bar{\lambda}(N(\boldsymbol{c}_1^{\sigma_z(i)}))+\bar{\lambda}(N(\bar{\boldsymbol{c}}_1^{\sigma_w(i)})) \\ -\bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)})\cap N(\boldsymbol{c}_1^{\sigma_z(i)}))-\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)})\cap N(\boldsymbol{c}_1^{\sigma_z(i)}))+i\mu \end{pmatrix}^{-1} \\ \geq &\sum_{\substack{\boldsymbol{a}\in\mathcal{P}(A),\boldsymbol{b}\in\mathcal{P}(B)\\ \boldsymbol{c}\in\mathcal{P}(B),\bar{\boldsymbol{c}}\in\mathcal{P}(\bar{C})\\ \sigma\in P(n,m,k,k)}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)}))+\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)}))+\bar{\lambda}(N(\boldsymbol{c}_1^{\sigma_z(i)}))+\bar{\lambda}(N(\bar{\boldsymbol{c}}_1^{\sigma_w(i)})) \\ -\bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)})\cap N(\boldsymbol{c}_1^{\sigma_z(i)}))-\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)})\cap N(\bar{\boldsymbol{c}}_1^{\sigma_w(i)}))+i\mu \end{pmatrix}^{-1}. \end{aligned} \tag{A.19}$$

Note that the only difference in the left and right sides of the last inequality are the terms $\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)}) \cap N(\bar{\boldsymbol{c}}_1^{\sigma_z(i)}))$ and $\bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)}) \cap N(\bar{\boldsymbol{c}}_1^{\sigma_w(i)}))$.

Equation A.19 can be expressed in the following equivalent way

$$\begin{aligned} &\mathrm{E}_{\mathfrak{S}\mathfrak{a}\mathfrak{b}\mathfrak{c}\bar{\mathfrak{c}}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}))+\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}))+\bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))+\bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})\cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))-\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})\cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))+i\mu \end{pmatrix}^{-1} \\ \geq\ &\mathrm{E}_{\mathfrak{S}\mathfrak{a}\mathfrak{b}\mathfrak{c}\bar{\mathfrak{c}}} \prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}))+\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}))+\bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))+\bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})\cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))-\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})\cap N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)}))+i\mu \end{pmatrix}^{-1}, \end{aligned} \tag{A.20}$$

where the expectation is over a uniformly random element $(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \bar{\mathfrak{c}}, \mathfrak{S})$ of the set $\mathcal{P}(A) \times \mathcal{P}(B) \times \mathcal{P}(C) \times \mathcal{P}(\bar{C}) \times P(n, m, k, k) = \mathcal{P}(A \cup B \cup C \cup \bar{C})$. In the following, we will simply write E in place of the symbol $\mathrm{E}_{\mathfrak{S}\mathfrak{a}\mathfrak{b}\mathfrak{c}\bar{\mathfrak{c}}}$.

To prove eq. A.20, we first prove it on a smaller σ-algebra. We say that two permutations γ_1 and $\gamma_2 \in \mathcal{P}(A \cup B \cup C \cup \bar{C})$ are equivalent if by dropping the overline marks of the particles in C in both γ_1 and γ_2, we obtain the same sequence of elements. Let $\mathcal{F}$ denote the σ-algebra generated by this equivalence relation. We show that for any $(\sigma, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{c}}) \in \mathcal{P}(A \cup B \cup C \cup \bar{C})$,

$$\begin{aligned} &\mathrm{E}\left[\prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}))+\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}))+\bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))+\bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})\cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))-\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})\cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))+i\mu \end{pmatrix}^{-1} \middle| \mathcal{F}\right](\sigma, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{c}}) \\ &\geq \mathrm{E}\left[\prod_{i=1}^{n+m+2k} \begin{pmatrix} \bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)}))+\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)}))+\bar{\lambda}(N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))+\bar{\lambda}(N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)})) \\ -\bar{\lambda}(N(\mathfrak{a}_1^{\mathfrak{S}_x(i)})\cap N(\mathfrak{c}_1^{\mathfrak{S}_z(i)}))-\bar{\lambda}(N(\mathfrak{b}_1^{\mathfrak{S}_y(i)})\cap N(\bar{\mathfrak{c}}_1^{\mathfrak{S}_w(i)}))+i\mu \end{pmatrix}^{-1} \middle| \mathcal{F}\right](\sigma, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{c}}). \end{aligned} \tag{A.21}$$

Fix $(\sigma, \boldsymbol{abc}\bar{\boldsymbol{c}}) \in \mathcal{P}(A \cup B \cup C \cup \bar{C})$. We can express $\bar{\boldsymbol{c}}$ as a composition of a permutation $\tau \in \mathcal{P}([k])$ and $\boldsymbol{c}$, so that $\bar{c}_i = c_{\tau(i)}$. Since all permutations in the equivalence class of $(\sigma, \boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \bar{\boldsymbol{c}})$ are equally likely, each conditional expectation in eq. A.21 can be expressed as an expectation over auxilliary i.i.d. Bernoulli(1/2) random variables $\{\beta_i\}_{i=1}^k$. To make this precise, let $S_{i,j} = \mathbb{1}(j \leq \sigma_z(i))$, $T_{i,j} = \mathbb{1}(\tau^{-1}(j) \leq \sigma_w(i))$, $p_i = i\mu + \bar{\lambda}(N(\boldsymbol{a}_1^{\sigma_x(i)})) + \bar{\lambda}(N(\boldsymbol{b}_1^{\sigma_y(i)}))$ and $\bar{\beta}_j = 1 - \beta_j$ for all $1 \leq i \leq n + m + 2k$ and $1 \leq j \leq k$. Also, let $\boldsymbol{c}^{\beta\sigma(i)} = \{c_j \in C : S_{i,j} = 1, \beta_j = 1\} \cup \{c_j \in C : T_{i,j} = 1, \beta_j = 0\}$ and similarly, $\boldsymbol{c}^{\bar{\beta}\sigma(i)} = \{c_j \in C : S_{i,j} = 1, \bar{\beta}_j = 1\} \cup \{c_j \in C : T_{i,j} = 1, \bar{\beta}_j = 0\}$. We have

$$\mathrm{E}_\beta\left[\prod_{i=1}^{n+m+2k}\left({p_i+\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)}))+\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})) \atop -\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)})\cap N(\boldsymbol{a}_1^{\sigma x(i)}))-\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)})\cap N(\boldsymbol{b}_1^{\sigma y(i)}))}\right)^{-1}\right.$$
$$\left.-\prod_{i=1}^{n+m+2k}\left({p_i+\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)}))+\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})) \atop -\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})\cap N(\boldsymbol{a}_1^{\sigma x(i)}))-\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})\cap N(\boldsymbol{b}_1^{\sigma y(i)}))}\right)^{-1}\right]\geq 0, \tag{A.22}$$

Using the fact that $\int_0^\infty e^{-cx}dx = \frac{1}{c}$ for any $c > 0$, we may write the above inequality as

$$\int_{\mathbb{R}^{+n+m+2k}}\mathrm{E}_\beta\left[\exp\left(-\sum_{i=1}^{n+m+2k}x_i\left({p_i+\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)}))+\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})) \atop -\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)})\cap N(\boldsymbol{a}_1^{\sigma x(i)}))-\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)})\cap N(\boldsymbol{b}_1^{\sigma y(i)}))}\right)\right)\right.$$
$$\left.-\exp\left(-\sum_{i=1}^{n+m+2k}x_i\left({p_i+\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)}))+\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})) \atop -\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})\cap N(\boldsymbol{a}_1^{\sigma x(i)}))-\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})\cap N(\boldsymbol{b}_1^{\sigma y(i)}))}\right)\right)\right]d\boldsymbol{x}_1^{n+m+2k}\geq 0. \tag{A.23}$$

It is enough to prove that the integrand in positive for every $\boldsymbol{x}_1^{n+m+2k}$. Symmetrizing the expression, by replacing β_j with $\bar\beta_j$, we obtain the following equivalent expression.

$$0\leq \mathrm{E}_\beta\left[\exp\left(-\sum_{i=1}^{n+m+2k}x_i\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)}))+x_i\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)}))\right)\right.$$
$$\times\left(\exp\Big(\textstyle\sum_i x_i\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)})\cap N(\boldsymbol{a}_1^{\sigma x(i)}))\Big)-\exp\Big(\sum_i x_i\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})\cap N(\boldsymbol{a}_1^{\sigma x(i)}))\Big)\right) \tag{A.24}$$
$$\left.\times\left(\exp\Big(\textstyle\sum_i x_i\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)})\cap N(\boldsymbol{b}_1^{\sigma y(i)}))\Big)-\exp\Big(\sum_i x_i\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)})\cap N(\boldsymbol{b}_1^{\sigma y(i)}))\Big)\right)\right].$$

To prove this, we use the FKG inequality on the lattice $\{0,1\}^k$ with measure

$$\nu(\boldsymbol{\beta})=\exp\left(-\sum_{i=1}^{n+m+2k}x_i\bar\lambda(N(\boldsymbol{c}^{\beta\sigma(i)}))+x_i\bar\lambda(N(\boldsymbol{c}^{\bar\beta\sigma(i)}))\right).$$

Claim 2. *The measure ν is log-submodular.*

Proof. Let $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \{0,1\}^{n+m+2k}$. Then,

$$
\begin{aligned}
&\begin{pmatrix} \bar{\lambda}(N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=1\}\cup\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=0\}))-\bar{\lambda}(N(\{c_j:S_{i,j}=1,\beta_j=1\}\cup\{c_j:T_{i,j}=1,\beta_j=0\})) \\ -\bar{\lambda}(N(\{c_j:S_{i,j}=1,\gamma_j=1\}\cup\{c_j:T_{i,j}=1,\gamma_j=0\}))+\bar{\lambda}(N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\}\cup\{c_j:T_{i,j}=1,\beta_j\wedge\gamma_j=0\})) \end{pmatrix} \\
&= \begin{pmatrix} \bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=1\})]-\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j=1\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\gamma_j=1\})]+\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})] \end{pmatrix} \\
&\qquad + \begin{pmatrix} \bar{\lambda}[N(\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=0\})]-\bar{\lambda}[N(\{c_j:T_{i,j}=1,\beta_j=0\})] \\ -\bar{\lambda}[N(\{c_j:T_{i,j}=1,\gamma_j=0\})]+\bar{\lambda}[N(\{c_j:T_{i,j}=1,\beta_j\wedge\gamma_j=0\})] \end{pmatrix} \\
&\qquad - \begin{pmatrix} \bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=0\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j=0\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\gamma_j=0\})] \\ +\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\wedge\gamma_j=0\})] \end{pmatrix}.
\end{aligned} \tag{A.25}
$$

Let us look at the first term in eq. A.25. We have

$$
\begin{aligned}
&\begin{pmatrix} \bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=1\})]-\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j=1\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\gamma_j=1\})]+\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})] \end{pmatrix} \\
&= \begin{pmatrix} \bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j=1\})\cap N(\{c_j:S_{i,j}=1,\gamma_j=1\})] \end{pmatrix},
\end{aligned}
$$

which is non-positive since $N(\{c_j : S_{i,j} = 1, \beta_j \wedge \gamma_j = 1\})$ is contained in both $N(\{c_j : S_{i,j} = 1, \beta_j = 1\})$ and $N(\{c_j : S_{i,j} = 1, \gamma_j = 1\})$.

Similarly, we may prove that the second term in eq. A.25 is non-positive. For the third term in that equation, we have:

$$
\begin{aligned}
&- \begin{pmatrix} -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\gamma_j=0\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j=0\})] \\ +\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=0\})] \\ +\bar{\lambda}N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\wedge\gamma_j=0\}) \end{pmatrix} \\
&\leq - \begin{pmatrix} -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=0\})] \\ -\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j=0\})] \\ +\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=0\})] \\ +\bar{\lambda}[N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=1\})\cap N(\{c_j:T_{i,j}=1,\beta_j\wedge\gamma_j=0\})] \end{pmatrix} \\
&\leq 0.
\end{aligned}
$$

By symmetry,

$$
\begin{pmatrix} \bar{\lambda}N(\{c_j:S_{i,j}=1,\beta_j\vee\gamma_j=0\}\cup\{c_j:T_{i,j}=1,\beta_j\vee\gamma_j=1\})-\bar{\lambda}N(\{c_j:S_{i,j}=1,\beta_j=0\}\cup\{c_j:T_{i,j}=1,\beta_j=1\}) \\ -\bar{\lambda}N(\{c_j:S_{i,j}=1,\gamma_j=0\}\cup\{c_j:T_{i,j}=1,\gamma_j=1\})+\bar{\lambda}N(\{c_j:S_{i,j}=1,\beta_j\wedge\gamma_j=0\}\cup\{c_j:T_{i,j}=1,\beta_j\wedge\gamma_j=1\}) \end{pmatrix} \leq 0.
$$

Therefore,

$$
\begin{aligned}
&\sum_i -x_i\bar{\lambda}(N(\boldsymbol{c}^{\beta\vee\gamma\sigma(i)})) - x_i\bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta\vee\gamma}\sigma(i)}) - \sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\beta\wedge\gamma\sigma(i)})) + x_i\bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta\wedge\gamma}\sigma(i)}) \\
&+\sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\beta\sigma(i)})) + x_i\bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta}\sigma(i)}) + \sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\gamma\sigma(i)})) + x_i\bar{\lambda}(N(\boldsymbol{c}^{\bar{\gamma}\sigma(i)}) \\
&\geq 0.
\end{aligned}
$$

Consequently, $\nu(\boldsymbol{\beta}\vee\boldsymbol{\gamma})\nu(\boldsymbol{\beta}\wedge\boldsymbol{\gamma}) \geq \nu(\boldsymbol{\beta})\nu(\boldsymbol{\gamma})$. ◻

Now we show that the two relevant functions in eq. A.24 are increasing in β.

Claim 3. *The functions*

$$
\begin{aligned}
f(\boldsymbol{\beta}) &= \exp\left(\sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\beta\sigma(i)}) \cap N(\boldsymbol{a}_1^{\sigma_x(i)}))\right) \\
&\quad - \exp\left(\sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta}\sigma(i)}) \cap N(\boldsymbol{a}_1^{\sigma_x(i)}))\right), \\
\textit{and } g(\boldsymbol{\beta}) &= \exp\left(\sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\beta\sigma(i)}) \cap N(\boldsymbol{b}_1^{\sigma_y(i)}))\right) \\
&\quad - \exp\left(\sum_i x_i\bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta}\sigma(i)}) \cap N(\boldsymbol{b}_1^{\sigma_y(i)}))\right)
\end{aligned}
$$

are increasing in β.

Proof. Let $h(\boldsymbol{\beta}) = \sum_i x_i\bar{\lambda}(N(c^{\beta\sigma(i)}) \cap N(\boldsymbol{a}_1^{\sigma_x(i)}))$. For any $J \subseteq [k]$, let $q_{i,J} = \bar{\lambda}(\cap_{j\in J} N(c_j) \cap N(\boldsymbol{a}_1^{\sigma_x(i)}))$. Using the inclusion-exclusion formula, we may write

$$
h(\boldsymbol{\beta}) = \sum_i x_i \sum_{J\subseteq[k]} (-1)^{|J|-1} q_{i,J} \prod_{j\in J}(\beta_j S_{i,j} + \bar{\beta}_j T_{i,j}).
$$

Now, let $\beta_1, \ldots, \beta_k$ be given. Fix $l \in [k]$. Fixing all β_j, $j \neq l$ and taking the difference of the values of h when $\beta_l = 1$ and $\beta_l = 0$, we obtain:

$$\begin{aligned}
&h(\boldsymbol{\beta}, \beta_l = 1) - h(\boldsymbol{\beta}, \beta_l = 0) \\
&= \sum_{i=1}^{n+m+2k} x_i (S_{i,l} - T_{i,l}) \sum_{J \subseteq [k-1]} (-1)^{|J|} q_{i,\{Jl\}} \prod_{j \in J} (\beta_j S_{i,j} + \bar{\beta}_j T_{i,j}) \\
&= \sum_{i=1}^{n+m+2k} x_i (S_{i,l} - T_{i,l}) \bar{\lambda}[(N(c_l) \cap N(\boldsymbol{a}_1^{\sigma_x(i)})) \backslash N(c_1^{\sigma_z(i)} \backslash c_l)] \\
&\geq 0,
\end{aligned}$$

since we have assumed that $S_{i,l} \geq T_{i,l}$. Similarly, taking

$$\begin{aligned}
h'(\boldsymbol{x}) &= \sum_i x_i \bar{\lambda}(N(c^{\beta\sigma(i)}) \cap N(\boldsymbol{a}_1^{\sigma_x(i)})) \\
&= \sum_i x_i \sum_{J \subseteq [k]} (-1)^{|J|-1} q_{i,J} \prod_{j \in J} (\bar{\beta}_j S_{i,j} + \beta_j T_{i,j}),
\end{aligned}$$

we have

$$\begin{aligned}
&h'(\boldsymbol{\beta}, \beta_l = 1) - h'(\boldsymbol{\beta}, \beta_l = 0) \\
&= \sum_{i=1}^{n+m+2k} x_i (T_{i,l} - S_{i,l}) \bar{\lambda}[(N(c_l) \cap N(\boldsymbol{a}_1^{\sigma_x(i)})) \backslash N(c_1^{\sigma_w(i)} \backslash c_l)] \\
&\leq 0.
\end{aligned}$$

Thus, $f(\boldsymbol{\beta}, \beta_l = 1) - f(\boldsymbol{\beta}, \beta_l = 0) \geq 0$. By symmetry in the problem, this is also true for g. $\square$

We are now in a position to apply the FKG theorem to the RHS of eq. A.24, and since

$$\begin{aligned}
&\mathrm{E}_\beta \Bigg[e^{-\sum_{i=1}^{n+m+2k} x_i \bar{\lambda}(N(\boldsymbol{c}^{\beta\sigma(i)})) + x_i \bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta}\sigma(i)}))} \\
&\quad \times \left(e^{\sum_i x_i \bar{\lambda}(N(\boldsymbol{c}^{\beta\sigma(i)}) \cap N(\boldsymbol{a}_1^{\sigma x(i)}))} - e^{\sum_i x_i \bar{\lambda}(N(\boldsymbol{c}^{\bar{\beta}\sigma(i)}) \cap N(\boldsymbol{a}_1^{\sigma x(i)}))} \right) \Bigg] \\
&= 0,
\end{aligned}$$

we obtain the result.

A.7 Table of Notation

D	Domain of interaction of particles. A metric space
$\boldsymbol{C} := \{\mathtt{R}, \mathtt{B}\}$	The set of types of particles, *reds* and *blues*
$\mathtt{u}$, $\mathtt{m}$	Marks to indicate whether a particle is matched or unmatched in the detailed processes
$\mathtt{u}, \mathtt{m}$	Marks to indicate whether a particle is matched or unmatched in the detailed processes
$\bar{D}$, $\hat{D}$	$\bar{D} := D \times C$, $\hat{D} := D \times C \times \{\mathtt{u}, \mathtt{m}\}$
$\bar{\mathtt{R}} := \mathtt{B}$, $\bar{\mathtt{B}} := \mathtt{R}$	Opposite color
λ	Radon measure on D
$m_{\boldsymbol{C}}$	Counting measure on $\boldsymbol{C}$
$\bar{\lambda}$, $\hat{\lambda}$	$\bar{\lambda} := \lambda \otimes m_{\boldsymbol{C}}$, $\hat{\lambda} := \lambda \otimes m_C \otimes m_C$ on $\bar{D}$.
μ	The parameter of the exponential random variables describing patience of particles.
$M(D, K)$	Space of simple Radon counting measures on D, with marks in K
$O(D, K)$	Space of simple locally-finite ordered subsets of D, with marks in K
$\lvert\gamma\rvert$, $\gamma \in O(D, K)$	Number of elements in γ
γ^x	The set $\{y \in \gamma : y <_\gamma x\}$ ordered as in γ
$p_x,\ b_x,\ c_x,\ w_x,\ x \in D \times \boldsymbol{C}$	Position, birth time, color and patience of x, i.e., $x = (p_x, c_x)$.
$N(A)$, $A \subset D \times \boldsymbol{C}$	$N(A) := \cup_{x \in A}\{y \in D \times \boldsymbol{C} : c_y \neq c_x, d(p_y, p_x) < 1\}$.
W_x	Region of maximum priority of $x \in \gamma$. $W_x = N(x) \backslash N(\gamma_x)$
$\eta_t \in O(D, \boldsymbol{C})$	Ordered collection of particles present in the system at time t
Φ	Poisson arrival process used in the construction of the process. It is a random element of $M(D \times \mathbb{R}^+, \boldsymbol{C} \times \mathbb{R}^+)$
S_t	Set of discrepancies $\eta_t^0 \triangle \eta_t^1$ in the CFTP construction
κ	Killing function

m	Matching function
$\hat{\eta}_t$	Backward detailed process
$\check{\eta}_t$	Forward detailed process
$Q^i_{\mathtt{u}}(\gamma)$	For $\gamma \in O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$, it is the number of unmatched particles among the first i particles of γ.
$Q^i_{\mathtt{m}}(\gamma)$	For $\gamma \in O(D, \boldsymbol{C} \times \{\mathtt{u}, \mathtt{m}\})$, it is the number of matched particles excluding the first i particles of γ.
$\hat{\pi}$	Density of the stationary measure of the Backward detailed process
π	Density of the stationary measure of the process η_t
$\tilde{\pi}$	Janossy density of stationary version of the point process η_0
$\mathcal{P}(C)$	Set of all permutations of the elements of a finite set C.
$P(m, n)$	The set of all paths in a square lattice from $(0, 0)$ to (m, n)
(σ, X^n_1, Y^m_1)	A representation of the map that gives the canonical bijection between $P(n, m) \times \mathcal{P}(x^n_1) \times \mathcal{P}(y^m_1)$ and $\mathcal{P}(x^n_1, y^m_1)$.

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